
Tissular coupling and frequency locking

II infinite population

Laurent Gaubert, Pascal Redou, and Jacques Tisseau

Centre Européen de Réalité Virtuelle, LISYC EA3883 UBO/ENIB, 25 rue Claude Chappe, 29280 Plouzané, France gaubert@enib.fr

1 Introduction

In an accompanying paper [1], we addressed the issue of synchronization under the framework of tissular coupling and in the case of a finite population. This issue is one of the main dynamical emergent property of complex systems. Indeed, this phenomenon actually occurs in natural systems at many scales (from cell to whole ecological systems), as it has been shown in [9, 10]. Many examples can be found in various fields of experimentation and theoretical research ([4, 6, 5, 11]). It is in particular a key concept in the field of non-linear systems' dynamics, especially through the notion of chaotic systems' synchronization [7, 3]. This wide source of examples explain why this field of research is highly interdisciplinary, from pure theory to concrete applications and experimentations. The classical concept of synchronization is related to the locking of the basic frequencies and instantaneous phases of regular oscillations. Those issues are usually addressed by studying specific coupled systems, using classical tools of the field (see for instance [8]). Convinced that synchronization phenomenon is completely natural in a large variety of coupled dynamical systems, we proposed in a preceding paper a new approach to the subject: we built a general model of coupled systems, called "tissular coupling". Within natural assumptions on interactions of those coupled systems, we proved that, for a wide class of autonomous differential systems, as soon as a finite population exhibits oscillating behaviors, their frequencies are mutually locked to a single value [1]. This paper presents a new piece of answer to those synchronization issues as it addresses the issue of an infinite compact connected population. Even if the general questions addressed in this paper are similar to those addressed in the later one, this study is done with completely different techniques than in the finite situation and the results achieved require assumptions that are specific to the continuous situation. This paper stresses the fact that synchronization is a very natural phenomenon that can be find in many kind of coupled dynamical systems.

In the first section, we recall some parts of the framework already introduced and some mathematical tools specific to our problem. Then we address a sub problem which deals with diffuse coupling, concluding with a general result on synchronization. Finally, we go back to the general case of a uniformly linear tissular coupling and show how those dynamical systems, and based on which assumptions, exhibits the frequencies locking property.

2 Basic material and notation

The main background of this work can be found in the accompanying paper [1] and in [2], so we just recall here some notations and concentrate on the new materials specific to the case of an infinite population.

- \mathcal{P} is a measured space with finite measure λ . Typical populations are finite populations with counting measure or measurable subset of \mathbb{R}^m with Lebesgue measure, so that \mathcal{P} has a topology. Every $p \in \mathcal{P}$ is called a cell;
- A **state** of \mathcal{P} is an element of $\mathcal{S} = \mathcal{M}_b(\mathcal{P}, \mathbb{R}^n)$, the space of measurable bounded applications from \mathcal{P} to \mathbb{R}^n . A state of \mathcal{P} is denoted s , $s(p)$ standing for the state of the cell p ;
- let I be a real interval and $r \in \mathbb{N}$. A **trajectory** of \mathcal{P} is an element of $\mathcal{F}(I, \mathcal{S})$. For concision purpose, any trajectory will be written s again, and $s(t, p)$ stands for the state of cell p at time t . Then, the space of trajectories is:

$$\mathcal{T}^r = \left\{ \mathcal{C}^r(I, \mathcal{S}), \forall \text{ compact } J \subset I, \sup_{J \times \mathcal{P}} \|s(t, p)\| < \infty \right\}$$

A trajectory s is regular only along the time t (excepted in terms of measurability), thus we use the following unambiguous notation :

$$s'(t, p) = \partial_1 s(t, p)$$

- All norms are denoted in the same manner :

$$s \in \mathcal{S} : \|s\| = \sup_{\mathcal{P}} \|s(p)\|$$

$$s \in \mathcal{T} : \|s\| = \sup_{I \times \mathcal{P}} \|s(p, t)\|$$

$$\|s(t, \cdot)\| = \sup_{\mathcal{P}} \|s(t, p)\|$$

$$\|s(\cdot, p)\| = \sup_I \|s(t, p)\|$$

- a **period on** \mathcal{P} is a map τ from \mathcal{P} to \mathbb{R}_+^* . A trajectory $s \in \mathcal{T}^r$ is said to be **τ -periodic** if for any $p \in \mathcal{P}$, $s(\cdot, p)$ is $\tau(p)$ -periodic. $\tau(p)$ is then called the

period of p . The space of such trajectories is written \mathcal{T}_τ^r . For any $s \in \mathcal{T}_\tau^0$ we define the τ -mean $I^\tau(s)$ of s as:

$$I^\tau(s)(p) = \frac{1}{\tau(p)} \int_0^{\tau(p)} s(t, p) dt$$

- If C is a coupler on \mathcal{P} and τ is a period on \mathcal{P} , the set $A(\tau, C)$ is defined as:

$$A(\tau, C) = \mathcal{T}^\infty \cup C^{-1}(\mathcal{T}^\infty)$$

In this paper, \mathcal{P} stands for an infinite compact connected population, with finite measure λ , mostly a measurable subset of \mathbb{R}^m with Lebesgue measure. Any period τ on \mathcal{P} is assumed to be measurable and bounded. On the contrary of the finite population case, to define a tissular coupler one needs to specify both its diffuse part c_d and its atomic part c_a . Such a coupler will be called **diffuse** if $c_a = 0$.

Partition associated to a periodic motion

Let suppose that s is a τ -periodic trajectory of \mathcal{P} , it divides \mathcal{P} into a partition according to the period of each cell. Let $p \in \mathcal{P}$ and $a \in \mathbb{R}^*$, we define the following application:

$$\tau_a(p) = a\tau(p)$$

and for any subset $A \subset \mathbb{R}$:

$$\mathcal{P}_p^A = \bigcup_{a \in A} \tau_a^{-1}(\tau(p))$$

In other words:

$$\mathcal{P}_p^A = \{q \in \mathcal{P} : \exists a \in A, a\tau(q) = \tau(p)\}$$

As soon as $1 \in A$, the set of all distinct \mathcal{P}_p^A realizes a partition of \mathcal{P} , we write \mathcal{P}^A this partition (with non-empty sets). The main examples are:

- if all the periods constitute a dependent set, then $\mathcal{P}^\mathbb{Q} = \{\mathcal{P}\}$;
- if all the periods are identical (\mathcal{P} is synchronized), then $\mathcal{P}^1 = \{\mathcal{P}\}$.

Measures associated to an uniformly linear coupling

The next tool we need is a method to measure the latter sets. But this cannot be done independently from the coupler itself. This is why we introduce the following family of measures on \mathcal{P} , one for each cell p :

$$\lambda_p(B) = \int_B \|c_d(p, q)\| dq$$

We recall that the support of a measure is defined to be the largest closed subset of \mathcal{P} for which every open neighborhood of every cell of the set has positive measure. Let S_p stands for the support of $c_d(p, \cdot)$, this support indicates where the measure λ_p lives and then which cells influence p in its evolution. In virtue of c_d 's continuity, we know that for any measurable subset B of \mathcal{P} , the following equivalence holds:

$$\lambda_p(B) > 0 \Leftrightarrow \lambda(B \cap S_p) > 0$$

Moreover, in the case of a diffuse coupler, this family of measures is uniformly absolutely continuous with respect to λ :

Lemma 1. *If C is an uniformly linear and diffuse coupler, then there exists a constant $m_C > 0$ such that for any measurable subset B of \mathcal{P} we have:*

$$\lambda_p(B) \leq m_C \lambda(B) \quad , \quad \forall p \in \mathcal{P}$$

Useful subsets of $A(\tau, C)$

The kind of systems we want to handle comes from differential equations studied near limit cycles, so, like in the finite population case, we need to avoid some degenerated trajectories. In order to do this, if $s \in \mathcal{T}^\tau$, we define the set of all cells on which the variations of s are not negligible. Precisely, if $\varepsilon > 0$, we define the following set:

$$V^\varepsilon(s) = \left\{ p \in \mathcal{P} \mid \inf_{c \in \mathbb{R}^n} \|s(\cdot, p) - c\| > \varepsilon \right\}$$

We call ε -kernel of C the set:

$$\text{Ker}^\varepsilon(C) = \{s \in \mathcal{T}^\tau, \quad V^\varepsilon(C(s)) \neq \mathcal{P}\}$$

An element in the ε -kernel of C contains, for at least one cell p , an application $s(\cdot, p)$ which image under C is uniformly near zero. Those solutions are out of interest here because they can be seen as solution of a system on a sub-population of C . Indeed for such a p $s'(\cdot, p)$ is negligible and $s(\cdot, p)$ nearly constant. In other words, we only consider trajectories which dynamic in each cell is "truly" periodic.

Given a period τ on \mathcal{P} , any element s of $A(\omega, C)$ and $\delta > 0$, the following set gathers cells which period are "not isolated", it will be naturally linked to $V^\varepsilon(s)$:

$$R_C^\delta(\tau) = \{p \in \mathcal{P} \mid \lambda_p(\mathcal{P}_p^\mathbb{Q}) > \delta \quad \}$$

If $p \in R_C^\delta(\tau)$, there is a set, which measure is at least δ , consisting of cells with periods commensurable with $\tau(p)$. Moreover, those cells are situated in the neighborhood of p defined by S_p .

Ergodic alternative

Like in the finite case, we will use this alternative within the following form [2]:

Proposition 1. *Let τ be a period on \mathcal{P} and C a tissular diffuse coupler, then for all s in $A(\tau, C)$ we have:*

$$C(s)(t, p) = \int_{\mathcal{P}_p^{\mathbb{Q}}} c_d(s(t, p), p, q) s(t, q) dq + \int_{\mathcal{P} - \mathcal{P}_p^{\mathbb{Q}}} c_d(s(t, p), p, q) \tau(q) I^{\tau(q)}(s)(q) dq$$

This results clearly shows that the ergodic theorem acts like a filter on periodic solutions, separating commensurable periods from the others. It is a key element of our proofs.

3 Synchronization with uniformly linear diffuse tissular coupler

As it is suggested in the title, in this section the kind of coupler we handle has the following shape:

$$C(s) = \int_{\mathcal{P}} c_d(., q) s(q) dq = L_C . s$$

Here is the first result explaining the link between the sets V and R . It will be generalized in the next result:

Proposition 2. *Let τ be a period on \mathcal{P} , for any s in $A(\tau, C)$ the following inclusion holds:*

$$V^0(C(s)) \subset R_C^0(\tau)$$

Noticing that the elements of $A(\tau, C)$ are uniformly bounded on I , we can then consider $\|s\| = \sup_{I \times \mathcal{P}} \|s(t, p)\|$ and use it to acquire the generalization:

Proposition 3. *Let τ be a period on \mathcal{P} , for any s in $A(\tau, C)$ we have:*

$$V^\varepsilon(C(s)) \subset R_C^{\frac{\varepsilon}{\|s\|}}(\tau)$$

In particular, as \mathcal{P} is a compact set, we can state:

Corollary 1. *Let τ be a period on \mathcal{P} and suppose there exists $\varepsilon > 0$ and $e \in A(\tau, C) - \text{Ker}^\varepsilon(C)$, then there exists a finite number of cells p_1, \dots, p_j such that:*

$$\mathcal{P} = \bigcup_{i=1}^j \mathcal{P}_{p_i}^{\mathbb{Q}}$$

The latter result leads us to this somewhat different result:

Corollary 2. *Let τ be a period on \mathcal{P} and suppose there exists $\varepsilon > 0$ and $e \in A(\tau, C) - \text{Ker}^\varepsilon(C)$, then there exists a sequence $\{p_i\}_{i \in \mathbb{N}}$ such that :*

$$\mathcal{P}_{p_i}^{\mathbb{Q}} = \bigcup_i \mathcal{P}_{p_i}^1$$

These results indicate a great reduction of the problem as they show that the periods cannot be arbitrary scattered. We are now able to prove the main result of this section. In order to handle only true cases of synchronization, we add an assumption on τ so that we won't have to deal with sub-period stuff:

Definition 1. *τ is said to be simple if $2 \text{Conv}(\tau(\mathcal{P})) \cap \text{Conv}(\tau(\mathcal{P})) = \emptyset$.*

Under this hypothesis, we have the following global synchronization result:

Theorem 1. *Suppose τ is a simple period on \mathcal{P} , if there exists $\varepsilon > 0$ and s in $A(\tau, C) - \text{Ker}^\varepsilon(C)$, then τ is a constant map on \mathcal{P} .*

Remark 1. To prove this result, we used the continuity of τ , deduced from the one of c_d . Nevertheless, leaving this assumption aside, we can get to an interesting result which could lead to generalization. We only need corollary 2 (we point that its validity does not directly depends on c_d 's continuity but on the uniform absolute continuity of the family λ_p and on the fact that $\|c_d(\cdot, \cdot)\|$ is uniformly bounded on \mathcal{P}^2).

Definition 2. *The set of point with isolated period, written \mathcal{P}^0 , is defined as:*

$$\mathcal{P}^0 = \{p \in \mathcal{P}, \quad \lambda_p(\mathcal{P}_p^1) = 0\}$$

The following proposition shows that under minimal assumptions, there are almost no points with isolated period:

Theorem 2. *Suppose there exists a partition as in the corollary 2 and that*

$$\mathcal{P} \subset \bigcup_{p \in \mathcal{P}} S_p$$

then $\lambda(\mathcal{P}^0) = 0$.

The latter result gives a global information that is verified in a general case: almost no cell has an isolated period. Those two results are really different, the first one is less theoretical but refers to stronger hypotheses than the second one, which comes under the field of measure theory.

4 Synchronization with uniformly linear tissular coupler

Now, we consider the general case, leaving aside the assumption that C is diffuse. Thus, in this last and short section, we consider a uniformly linear tissular coupler on \mathcal{P} (c_a needs no more to be zero). Like in the finite population case, the first step consists in detecting and eliminating degenerated solutions:

Definition 3. Let $s \in \mathcal{T}(I, \mathcal{E}_b)$ and $\varepsilon \geq 0$, s is said to be an ε -eigenvector (for C) if:

$$C(s) - c_a \in \text{Ker}^\varepsilon(C)$$

An ε -eigenvector is therefore a s which contains a map $e(\cdot, p)$ uniformly near a hypothetical eigenvector of C for the eigenvalue $c_a(p)$.

Finally, here is the generalization of theorem 1:

Theorem 3. Let τ be a simple period on \mathcal{P} and $\varepsilon > 0$. If there exists $s \in A(\tau, C)$ which is not an ε -eigenvector, then τ is a constant map.

5 Conclusion

In this work, we have given another example of how the notion of tissular coupling can help to address synchronization issues within a wide variety of coupled systems. Our approach is completing the classical methods showing that within suitable conditions, synchronization must occur along time. Those results are often qualitatively dependent to the studied systems. After the first positive result of our accompanying paper [1], we hold to our approach and didn't prove that synchronization ultimately happens but demonstrated that if coupled systems oscillate, then they must be synchronized. As in the finite population situation, the loss in time evolution informations is compensated by very general results, almost independent from the individual differential systems to be coupled. Furthermore, conclusions are less restrictive than in [1] (there is no threshold like "half of the population is synchronized"). Moreover, we hope that this complementary approach to synchronization issues will lead to future developments as it brings about, also in this paper, many mathematical tools unfamiliar to this field of research. Furthermore, other examples of the use of tissular coupling can be found in [2] concerning emergence of spatial patterns. All those works and results are inspiring and seem to prove that tissular coupling will be a prolific framework, completing what already exists.

References

- [1] Laurent Gaubert. Tissular coupling and frequency locking: I finite population.

- [2] Laurent Gaubert. *Auto-organisation et émergence dans les systèmes couplés, individuation de données issues de systèmes biologiques couplés*. PhD thesis, Université de Bretagne Occidentale, 2007.
- [3] J.-P. Goedgebuer, P. Levy, and L. Larger. Laser cryptography by optical chaos. In J.-P. Goedgebuer, N.N. Rozanov, S.K. Turitsyn and A.S. Akhmanov, and V.Y. Panchenko, editors, *Optical Information, Data Processing and Storage, and Laser Communication Technologies*, volume 5135, pages 14–20, 2003.
- [4] Didier Gonze, Jos Halloy, and Albert Goldbeter. Stochastic models for circadian oscillations: Emergence of a biological rhythm. *International Journal of Quantum Chemistry*, 98:228–238, 2004.
- [5] Frank C. Hoppensteadt and Joseph B. Keller. Synchronization of periodical cicada emergences. *Science*, 194(4262):335–337, Oct 1976.
- [6] Donald C. Michaels, Edward P. Matyas, and Jose Jalife. Mechanisms of sinoatrial pacemaker synchronization: A new hypothesis. *Circ. Res.*, 61:704–714, 1987.
- [7] Louis M. Pecora, Thomas L. Carroll, Gregg A. Johnson, Douglas J. Mar, and James F. Heagy. Fundamentals of synchronization in chaotic systems, concepts, and applications. *Chaos*, 7(4):520–543, 1997.
- [8] Arkady Pikovsky, Michael Rosenblum, and Jürgen Kurths. *Synchronization: A Universal Concept in Nonlinear Sciences*. Cambridge University Press, 2001.
- [9] Ilya Prigogine. *Introduction to Thermodynamics of Irreversible Processes*. Wiley, New York, 1967.
- [10] Ilya Prigogine. *Theoretical Physics and Biology*. M Marois, North-Holland: Amsterdam, 1969.
- [11] Kurt Wiesenfeld, Pere Colet, and Steven H Strogatz. Synchronization transitions in a disordered josephson series array. *Phys. Rev. Lett.*, 76:404–407, 1996.