Complex systems representation by C^k spline functions

Youssef Slamani , Marc Rouff and Jean Marie Dequen

Laboratoire Universitaire des Sciences Appliquées de Cherbourg
Université de CAEN Basse Normandie
BP-78,F-50130 Cherbourg-Octeville, France
y.slamani.lusac@chbg.unicaen.fr

Abstract

This work presents the principal algebraic, arithmetic and geometrical properties of the C^k spline functions as well in the temporal space as in the frequencies space. Thanks to their good properties of regularity, of smoothness and compactness in both spaces, precise and powerful computations implying C^k spline functions can be considered.

The main property of C^k spline functions is to have for coefficients of their functional expansion of a considered function, the whole set of partial or complete derivatives up to the order k of the considered function.

This fundamental property allows a much easier representation of complex systems as well in the linear case as in the nonlinear case. Then traditional differential and integral calculations leads in the space C^k spline functions space to new functional and invariant calculations.

Key Words

Fourier transforms, C^k spline functions, C^k wavelets, interpolation, linear system, nonlinear system, functional analysis.

1. Introduction

The C^k spline functions appear now as a powerful tool, not only for numerical computation, but also for the formal calculation. The basic property of these functions is to have for coefficients of functional development of a considered function, the whole partial or complete derivatives of this function until the order k.

Thus the C^k spline functions expansion can be view as interpolating functions of local Taylor-Maclaurin expansions up to the order k defined at each point of discretization of the considered function whereas the known traditional splines (B splines....) are in general preset functions. This property which is specific of the C^k spline functions leads to a uniform convergence of the solution in function of k and the number of point of dicretization. This uniform convergence is to our knowledge only seen in C^k spline functions computations

Moreover, the Fourier transforms of these C^k splines are C^k wavelets which have the same remarkable property to have for coefficients of the spectrum functional development of a function in their space the partial or complete derivatives of the considered function until the order k i.e. the same ones as those obtained in the temporal C^k splines space. This fact opens the way, for example, to a new time-frequency signal analysis and especially to a new functional and invariant calculation replacing the differential and integral calculus by simple index manipulation. This allows to simplify representation of complex system as well linear as non linear.

An other important advantage of these functions (splines or wavelets) is their remarkable property to obtain a functional representation with a very high accuracy with a small number of coefficients, and this in both spaces with the same precision.

In this article, first we present the main properties of C^k Spline functions in the direct space.

In a second part, we gather the properties of C^k Splines spectra or C^k wavelets.

In the third part, an example of C^k wavelet expansion applied to sin(x) is shown and provides the uniform convergence with excellent rate and accuracy. Only one dimensional C^k spline functions are considered but they can be easily extended to multivariate case.

Finally in the fourth part, we show that the representation of the state equations of as well linear systems as nonlinear systems leads, in the space of C^k splines, to a functional and invariant calculation at place of the classical differential and integral calculus.

2. C^k Spline functions. Temporal space

2.1 The equally spaced nodes representation

Let u(x) be a k time continuous and differentiable one-dimensional function defined on an appropriate set Ω, which contains I+1 equally spaces nodes $x_0, x_1, \ldots, x_I$.

$u_{k,I}(x)$ the C^k approximation of this function on Ω is written as,

$$u_{k,I}(x) = \sum_{\nu=0}^{k} \left\{ \sum_{i=1}^{I-1} \left(u_i^{\nu} S_i^{k,\nu}(x) \right) + u_0^{\nu} P_{R0}^{k,\nu}(x) + u_I^{\nu} P_{LI}^{k,\nu}(x) \right\} \quad (1)$$

where u_i^{ν} is the ν^{th} derivate of u(x) with respect to x at the node x_i, $S_i^{k,\nu}(x)$ is the ν^{th} C^k Spline function centered at the node x_i and defined on the set $[x_{(i-1)}, x_{(i+1)}]$, $P_{Ri}^{k,\nu}(x)$ and $P_{Li}^{k,\nu}(x)$ are respectively the right and left sides of the C^k Spline function $S_i^{k,\nu}(x)$ and are defined respectively on the intervals $[x_i, x_{(i+1)}]$ and $[x_{(i-1)}, x_i]$. These definitions are done $\forall i \in, [0, I]$ $\forall \nu \in [0, k]$.

2.2 The unequally spaced nodes representation

Under an arbitrary nodes distribution on the set Ω, i.e. I+1 unequally spaced nodes $x_0, x_1, \ldots, x_I$, $u_{k,I}(x)$ can be written as,

$$u_{k,I}(x) = \sum_{\nu=0}^{k} \sum_{i=1}^{I} \left(u_{i-1}^{\nu} P_{R(i-1)}^{k,\nu}(x) + u_i^{\nu} P_{Li}^{k,\nu}(x) \right) \quad (2)$$

where $P_{R(i-1)}^{k,\nu}(x)$ and $P_{Li}^{k,\nu}(x)$ have the same support $[x_{(i-1)}, x_i]$.

2.3 Algebraic properties

Referring to [1], the C^k Spline functions are defined by,

$$\frac{d^l}{d\,x^l} S_i^{k,\nu}(x) \Big|_{x=x_j} \equiv \delta[i-j]\,\delta[\nu-l]$$

witch leads to,

$$S_i^{k,\nu}(x) = \sum_{d=0}^{2k+1} a_d^{\nu} \left[\frac{x-x_i}{\Delta x_{(i+1)}} \right]^d, \ \text{on}[x_{i-1}, x_{(i+1)}] \quad (3),$$

$$P_{Ri}^{k,\nu}(x) = \sum_{d=0}^{2k+1} a_{Rd}^{\nu} \left[\frac{x-x_i}{\Delta x_{(i+1)}} \right]^d, \ \text{on}[x_i, x_{(i+1)}] \quad (4),$$

$$P_{Li}^{k,\nu}(x) = \sum_{d=0}^{2k+1} a_{Ld}^{\nu} \left[\frac{x-x_i}{\Delta x_i} \right]^d, \ \text{on}[x_{(i-1)}, x_i] \quad (5),$$

And,

$$S_i^{k,\nu}(x) = P_{Ri}^{k,\nu}(x) + P_{Li}^{k,\nu}(x), \ \text{on}[x_{(i-1)}, x_{(i+1)}], \quad (6)$$

where the a_d^{ν} or the a_{Rd}^{ν} and a_{Ld}^{ν} are the polynomial coefficients generating the C^k Spline functions and $\Delta x_i = x_i - x_{(i-1)}$. The main properties of these coefficients are,

- $\left|a_{Ld}^{\nu}\right| = \left|a_{Rd}^{\nu}\right| \equiv \left|a_d^{\nu}\right|, \forall \nu \in [0,k], \forall d \in [0, 2k+1]$
- $a_{Ld}^{\nu} = a_{Rd}^{\nu} \equiv a_d^{\nu}$ and

$$a_d^{\nu} = \delta(d-\nu)\frac{\Delta x^{\nu}}{\nu!}, \ \forall \nu \in [0,k], \forall d \in [0,k]$$

- $a_{Rd}^{\nu} = (-1)^{d-k} \left|a_{Ld}^{\nu}\right|, \forall \nu \in [0,k], \forall d \in [k+1, 2k+1]$ •
- $a_{Ld}^{\nu} = a_{Ld}^{\nu}\Big|_{\Delta x=1} * \Delta x^{\nu}$ and $a_{Rd}^{\nu} = a_{Rd}^{\nu}\Big|_{\Delta x=1} * \Delta x^{\nu} \quad (7)$

The a_{Ld}^{ν}, a_{Rd}^{ν}, *AND* a_d^{ν} $\in$ ¤

For example, the following list gives the a_{Rd}^{ν} and the a_{Ld}^{ν} for k=2 and $\Delta x = 1$.

$a_{R0}^0 = 1$ $a_{R0}^1 = 0$ $a_{R0}^2 = 0$ $a_{L0}^0 = 1$ $a_{L0}^1 = 0$ $a_{L0}^2 = 0$

$a_{R1}^0 = 0$ $a_{R1}^1 = 1$ $a_{R1}^2 = 0$ $a_{L1}^0 = 0$ $a_{L1}^1 = 1$ $a_{L1}^2 = 0$

$a_{R2}^0 = 0$ $a_{R2}^1 = 0$ $a_{R2}^2 = \frac{1}{2}$ $a_{L2}^0 = 0$ $a_{L2}^1 = 0$ $a_{L2}^2 = \frac{1}{2}$

$a_{R3}^0 = -10$ $a_{R3}^1 = -6$ $a_{R3}^2 = -\frac{3}{2}$ $a_{L3}^0 = 10$ $a_{L3}^1 = -6$ $a_{L3}^2 = \frac{3}{2}$

$a_{R4}^0 = -15$ $a_{R4}^1 = 8$ $a_{R4}^2 = \frac{3}{2}$ $a_{L4}^0 = 15$ $a_{L4}^1 = -8$ $a_{L4}^2 = \frac{3}{2}$

$a_{R5}^0 = -6$ $a_{R5}^1 = -3$ $a_{R5}^2 = -\frac{1}{2}$ $a_{L5}^0 = 6$ $a_{L5}^1 = -3$ $a_{L5}^2 = \frac{1}{2}$

2.4 Representation of the C^k Spline functions

Let us consider using Figure 1 the four $S_0^{3,\nu}(x)$, $\nu \in [0,3] \cap ¥$, $x \in [-1,1] \cap ¡$ centered at the point $x_0 = 0$.

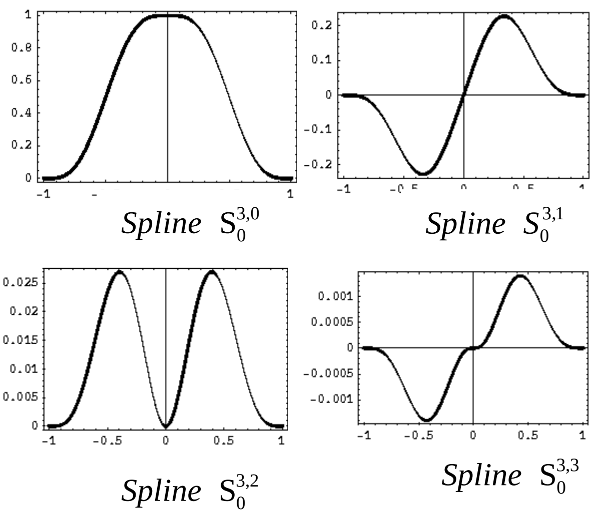


Figure1: *The Four* $S_0^{3,\nu}$, $\nu \in [0,3]$

We can see easily that

$$\left.\frac{d^l}{d\,x^l} S_0^{3,\nu}(x)\right|_{x=x_j} \equiv \delta[j-0]\,\delta[\nu - l]$$

i.e. each spline function assumes the representation of a fixed derivative 1 at the point O and has a zero derivative for $\nu \neq i$ and for $x_j \neq 0$ (orthogonality).

On the figure 1, the $P_{R0}^{3,\nu}(x)$ and the $P_{L0}^{3,\nu}(x)$ can be defined as respectively the right and the left sides of the $S_0^{3,\nu}(x)$ i.e. as the polynomial functions defined respectively on $[0,1]$ and $[-1,0]$ for each $S_0^{3,\nu}(x)$.

3. C^k Spline Spectra. Frequency space

Let u(x) be a k time continuous and differentiable one-dimensional function defined on an appropriate set Ω, which contains I+1 nodes $x_0, x_1, \ldots, x_I$.

Defining $\overline{u}_{k,I}(\theta)$, the Fourier Transform of $u_{k,I}(x)$ where θ is the dual Fourier variable of x, we have,

$$\overline{u}_{k,I}(\theta) = \int_0^X u_{k,I}(x) e^{-i\theta x} dx \quad (8)$$

3.1 The equally spaced nodes spectra interpolation

When the I+1 nodes $x_0, x_1, \ldots, x_I$ of the set Ω are equally spaced, we can replace $u_{k,I}(x)$ by (1), we obtain,

$$\overline{u}_{k,I}(\theta) = \sum_{\nu=0}^{k} \left\{ \sum_{n=1}^{I-1} \left(u_n^\nu \int_0^X S_n^{k,\nu}(x) e^{-i\theta x} dx \right) + u_0^\nu \int_0^X P_{R0}^{k,\nu}(x) e^{-i\theta x} dx + u_I^\nu \int_0^X P_{LI}^{k,\nu}(x) e^{-i\theta x} dx \right\}$$

Therefore,

$$\overline{u}_{k,I}(\theta) = \sum_{\nu=0}^{k} \left(\sum_{n=1}^{I-1} \left(u_n^\nu \, \overline{S}_n^{k,\nu}(\theta) \right) + u_0^\nu \, \overline{P}_{R0}^{k,\nu}(\theta) + u_I^\nu \, \overline{P}_{LI}^{k,\nu}(\theta) \right)$$

where the $\overline{S}_n^{k,\nu}(\theta)$, $\overline{P}_{R0}^{k,\nu}(\theta)$,and $\overline{P}_{LI}^{k,\nu}(\theta)$ are respectively the Fourier Transform of $S_n^{k,\nu}(x)$, $P_{R0}^{k,\nu}(x)$ and $P_{LI}^{k,\nu}(x)$.

Finally,

$$\overline{u}_{k,I}(\theta) = \sum_{\nu=0}^{k} \left(\sum_{n=1}^{I-1} \left(u_n^\nu \, \overline{S}_0^{k,\nu}(\theta) e^{-in\Delta x\theta} \right) + u_0^\nu \, \overline{P}_{R0}^{k,\nu}(\theta) + u_I^\nu \, \overline{P}_{L0}^{k,\nu}(\theta) e^{-iI\Delta x\theta} \right) \quad (9)$$

where the $\overline{S}_0^{k,\nu}(\theta)$, $\overline{P}_{R0}^{k,\nu}(\theta)$,and $\overline{P}_{L0}^{k,\nu}(\theta)$ are respectively the Fourier Transform of $S_0^{k,\nu}(x)$, $P_{R0}^{k,\nu}(x)$ and $P_{L0}^{k,\nu}(x)$ defined at the node x_0 and $\Delta x = x_1 - x_0 = x_2 - x_1 = \cdots = x_I - x_{I-1}$.

3.2 The unequally spaced nodes spectra interpolation

Under an arbitrary nodes distribution i.e. unequally spaced nodes $x_0, x_1, \ldots, x_I$ by the same way than in 3.1, $\overline{u}_{k,I}(\theta)$ can be written as,

$$\overline{u}_{k,I}(\theta) = \sum_{\nu=0}^{k} \sum_{i=1}^{I} \left(u_{i-1}^\nu \, \overline{P}_{R(i-1)}^{k,\nu}(\theta) + u_i^\nu \, \overline{P}_{Li}^{k,\nu}(\theta) \right) \quad (10)$$

3.3 C^k Spline Spectra

As we saw in section 3.1 and 3.2, calculating $\overline{u}_{k,I}(\theta)$ the Fourier Transform of $u_{k,I}(x)$, leads to study both the $\overline{S}_0^{k,\nu}(\theta)$, $\overline{P}_{Ri}^{k,\nu}(\theta)$, and $\overline{P}_{Li}^{k,\nu}(\theta)$ respectively the Fourier Transform of $S_0^{k,\nu}(x)$, $P_{Ri}^{k,\nu}(x)$ and $P_{Li}^{k,\nu}(x)$.

3.3.1 C^k Spline $\overline{S}_0^{k,\nu}(\theta)$ relations

The following Lemmas result from C^k Spline properties section 2-3 and from classical algebraic computations.

LEMMA 3.3.1.1 : Let $\overline{S}_0^{k,\nu}(\theta)$ be the Fourier transform of $S_0^{k,\nu}(x)$, we have,

for ν even,

$$\overline{S}_0^{k,\nu}(\theta)=2(-1)^{k+1}\Delta x^{\nu+1}\left\{\sum_{\ell=1}^{2k+2}\frac{\cos(\theta+\ell\pi/2)}{\theta^\ell}\alpha[\ell,\nu]-\sum_{j=E[\frac{k+1}{2}]}^{k}\frac{a_{2j+1}^{\nu}(2j+1)!(-1)^j}{a^{2\ell+2}}\right\}$$

and for ν odd,

$$\overline{S}_0^{k,\nu}(\theta)=2i(-1)^{k}\Delta x^{\nu+1}\left\{\sum_{\ell=1}^{2k+1}\frac{\sin(\theta+\ell\pi/2)}{\theta^\ell}\alpha[\ell,\nu]-\sum_{j=E[\frac{k}{2}]+1}^{k}\frac{a_{2j}^{\nu}(2j)!(-1)^j}{a^{2\ell+1}}\right.$$

with,

$$\alpha[\ell,\nu]=\sum_{j=k+1+max[\ell-k-2,0]}^{2k+1}\frac{a_j^{\nu}\ j!(-1)^j}{(j+1-\ell)!}+\frac{(-1)^k U[\nu+1-\ell]}{(\nu+1-\ell)!}$$

where $U[.]$, max(.,.) and E[.] are respectively the Heaviside, the maximum and the Floor functions. i is defined as $i^2=-1$.

LEMMA 3.3.1.2: $\overline{S}_0^{k,\nu}$ defined by the lemma 3.3.1.1 is singular at $\theta=0$. Near this point we have for ν even the following Taylor development,

$$\overline{S}_0^{k,\nu}(\theta)=(-1)^k 2\Delta x^{\nu+1}\sum_{\ell=0}^{+\infty}(-1)^\ell\theta^{2\ell}\beta_e[\ell,\nu]$$

with,

$$\beta_e[\ell,\nu]=\frac{(-1)^k a_\nu^\nu(2\ell+\nu)!}{(2\ell+\nu+1)!(2\ell)!}+\sum_{j=E[\frac{k}{2}]+1}^{k}\frac{a_{2j}^{\nu}(2j+2\ell)!}{(2j+2\ell+1)!(2\ell)!}-\sum_{j=E[\frac{k+1}{2}]}^{k}\frac{a_{2j+1}^{\nu}(2j+2\ell+1)!}{(2j+2\ell+2)!(2\ell)!}$$

for ν odd we have,

$$\overline{S}_0^{k,\nu}(\theta)=i(-1)^{k+1}2\Delta x^{\nu+1}\sum_{\ell=0}^{+\infty}(-1)^\ell\theta^{2\ell+1}\beta_o[\ell,\nu]$$

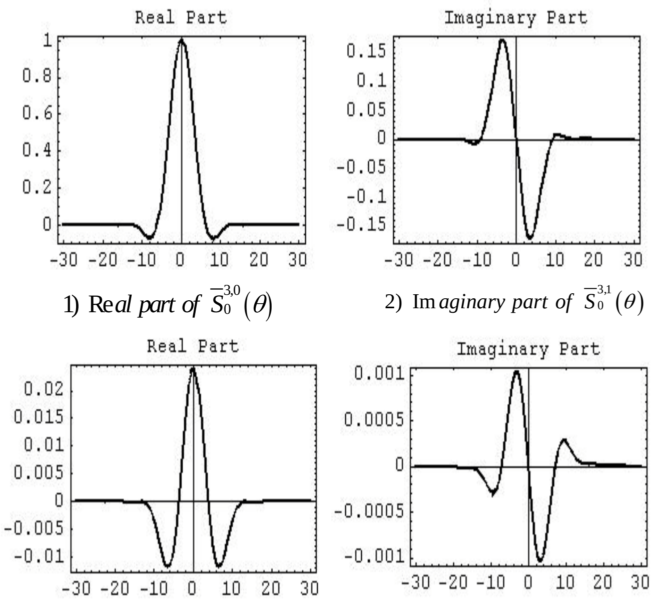
with,

$$\beta_o[\ell,\nu]=\frac{(-1)^k a_\nu^\nu(2\ell+\nu+1)!}{(2\ell+\nu+2)!(2\ell+1)!}+\sum_{j=E[\frac{k}{2}]+1}^{k}\frac{a_{2j}^{\nu}(2j+2\ell+1)!}{(2j+2\ell+2)!(2\ell+1)!}-\sum_{j=E[\frac{k+1}{2}]}^{k}\frac{a_{2j+1}^{\nu}(2j+2\ell+2)!}{(2j+2\ell+3)!(2\ell+1)!}$$

Clearly $\beta_e[\ell,\nu]$ and $\beta_o[\ell,\nu]$ ∈ ¤ , $\forall[\ell,\nu]\in[1,2k+2]\times[0,k]$.

3.3.2 Representations of $\overline{S}_0^{k,\nu}(\theta)$

The Fourier transform of these C^k Spline functions are Ck wavelets, and are given Figure 2 for k=3.The $\overline{S}_i^{k,\nu}(\theta)$ are real functions for v even and imaginary functions for v odd.



1) Real part of $\overline{S}_0^{3,0}(\theta)$ 2) Imaginary part of $\overline{S}_0^{3,1}(\theta)$

3) Real part of $\overline{S}_0^{3,2}(\theta)$ 4) Imaginary part of $\overline{S}_0^{3,3}(\theta)$

Figure 2: The four $\overline{S}_0^{3,\nu}(\theta)$, $\nu\in[0,3]$

We can notice their excellent localization properties.

3.3.3 $\overline{P}_{Ri}^{k,\nu}(\theta)$ and $\overline{P}_{Li}^{k,\nu}(\theta)$ relations

In this case, we define in Ω a series of intervals $\Delta x_i = x_{0i}-x_{0(i-1)}$. Same considerations as in paragraph 3.3.1.1. lead to the following lemmas:

Lemma 3.3.3.1: Let $\overline{P}_{Ri}^{k,\nu}(\theta)$ be the Fourier transform of $P_{Ri}^{k,\nu}(x)$, for ν even we have,

$$\overline{P}_{Ri}^{k,\nu}(\theta)=(-1)^k \Delta x_i^{\nu+1}\left\{i\left(\frac{(-1)^{k-\frac{\nu}{2}+1}}{\theta^{\nu+1}}-\sum_{d=E[\frac{k}{2}]}^{k}\frac{a_{2d}^{\nu}(2d)!(-1)^d}{\theta^{2d+1}}+\sum_{\ell=1}^{2k+2}\frac{sin(\theta+\ell\pi/2)}{\theta^{\ell}}\gamma[\ell]\right)+\left(\sum_{d=E[\frac{k}{2}]}^{k}\frac{a_{2d+1}^{\nu}(2d+1)!(-1)^d}{\theta^{2d+2}}-\sum_{\ell=1}^{2k+2}\frac{cos(\theta+\ell\pi/2)}{\theta^{\ell}}\gamma[\ell]\right)\right\}e^{-\theta i}$$

For ν odd we have,

$$\overline{P}_{Ri}^{k,\nu}(\theta)=(-1)^k \Delta x_i^{\nu+1}\left\{\left(\frac{(-1)^{k-(\frac{\nu+1}{2})}}{\theta^{\nu+1}}+\sum_{d=E[\frac{K+1}{2}]}^{k}\frac{(2d+1)!a_{(2d+1)}^{\nu}(-1)^{d+1}}{\theta^{2d+2}}+\sum_{\ell=1}^{2k+2}\frac{cos(\theta+\ell\pi/2)}{\theta^{\ell}}\gamma[\ell]\right)+i\left(\sum_{d=E[\frac{k}{2}]+1}^{k}\frac{a_{(2d)}^{\nu}(2d)!(-1)^d}{\theta^{2d+1}}-\sum_{\ell=1}^{2k+2}\frac{sin(\theta+\ell\pi/2)}{\theta^{\ell}}\gamma[\ell]\right)\right\}e^{-\theta i}$$

with,

$$\gamma[\ell]=\sum_{j=k+1+Max[\ell-(k+2),0]}^{2k+2}\frac{a_j^{\nu}j!(-1)^j}{(j+1-\ell)!}+\frac{(-1)^k U[\nu+1-\ell]}{(\nu+1-\ell)}$$

The various coefficients of these functions belong clearly to ¤ .

Lemma 3.3.3.2: $\overline{P}_{Ri}^{k,\nu}(\theta)$ is singular at $\theta=0$. Near this point, we have for all ν the following Taylor development,

$$\overline{P}_{Ri}^{k,\nu}(\theta)=\Delta x_i^{\nu+1}\left\{\left(\sum_{\ell=0}^{+\infty}\theta^{2\ell}\delta_r[\ell]\right)-i\left(\sum_{\ell=0}^{+\infty}\theta^{2\ell+1}\delta_i[\ell]\right)\right\}$$

with,

$$\delta_r[\ell]=\frac{a_{\nu}^{\nu}(-1)^{\ell}(2\ell+\nu)!}{(2\ell+\nu+1)!(2\ell)!}+\sum_{d=k+1}^{2k+1}\frac{a_d^{\nu}(-1)^{d+k+\ell}(2\ell+d)!}{(2\ell+d+1)!(2\ell)!}$$

$$\delta_i[\ell]=\frac{a_{\nu}^{\nu}(-1)^{\ell}(2\ell+\nu+1)!}{(2\ell+\nu+2)!(2\ell+1)!}+\sum_{d=k+1}^{2k+1}\frac{a_d^{\nu}(-1)^{d+k+\ell}(2\ell+d+1)!}{(2\ell+d+2)!(2\ell+1)!}$$

The $\delta_r[\ell]$ and the $\delta_i[\ell]$ belong to ¤
.

Lemma 3.3.3.3: Let $\overline{P}_{Li}^{k,\nu}(\theta)$ be the Fourier transform of $P_{Li}^{k,\nu}(x)$,

For ν even we have,

$$\Re e[\overline{P}_{Li}^{k,\nu}(\theta)e^{\theta i}]=\Re e[\overline{P}_{Ri}^{k,\nu}(\theta)e^{\theta i}]$$

and,

$$\Im m[\overline{P}_{Li}^{k,\nu}(\theta)e^{\theta i}]=-\Im m[\overline{P}_{Ri}^{k,\nu}(\theta)e^{\theta i}]$$

For ν odd we have,

$$\Re e[\overline{P}_{Li}^{k,\nu}(\theta)e^{\theta i}]=-\Re e[\overline{P}_{Ri}^{k,\nu}(\theta)e^{\theta i}]$$

and,

$$\Im m[\overline{P}_{Li}^{k,\nu}(\theta)e^{\theta i}]=\Im m[\overline{P}_{Ri}^{k,\nu}(\theta)e^{\theta i}]$$

Then as in the lemma 3.3.2.1., the various coefficients of these functions belong also to ¤ .

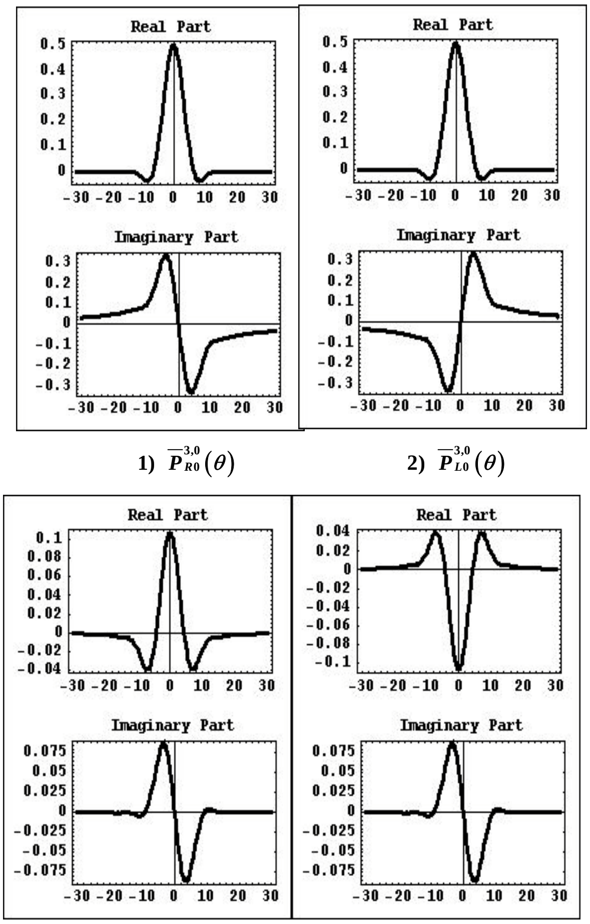


Figure 3: ***The real and imaginary parts of*** $\overline{P}_{R0}^{3,0}(\theta)$ ***and*** $\overline{P}_{L0}^{3,0}(\theta)$***. The other*** $\overline{P}_{R0}^{3,\nu}(\theta)$ ***and*** $\overline{P}_{L0}^{3,\nu}(\theta)$, $\nu\in[2,3]$ ***are not represented.***

3.3.4 Representation of the $\overline{P}_{Ri}^{k,\nu}(\theta)$ and the $\overline{P}_{Li}^{k,\nu}(\theta)$

The real parts and imaginary parts of spectra

$\overline{P}_{Ri}^{k,\nu}(\theta)$ and $\overline{P}_{Li}^{k,\nu}(\theta)$ for k=3 and $\Delta x = 1$ are shown in Figure 3.We can easily verify properties 3.3.2.3 and

4. Example: C^k wavelets representation of Sinus.

Figure 4 gives, left, the exact representation of the Fourier transform of Sin(x) on $\Omega = [0,10]$ and right, its wavelets representation for k=5 , I=5 and with equally spaced nodes ($\Delta x = \frac{5}{2}$).The wavelets interpolation of sin(x) is done with the use of (9).

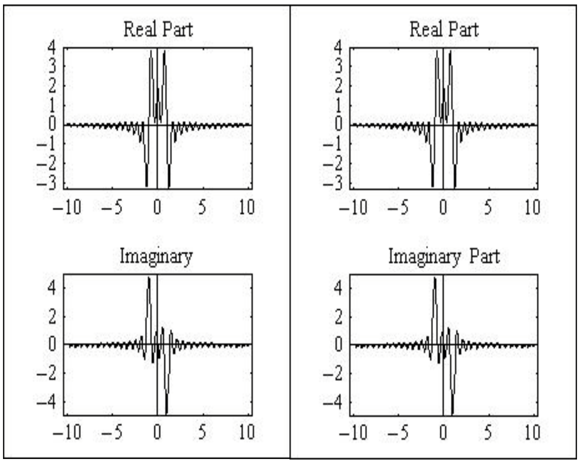


***Figure 4**: Exact Fourier Transform of* $\sin(x)$ $x \in [0,10]$ *(left) and its* C^k *wavelet representation (right)*

We can notice the very good precision of the C^k wavelets representation for a small numbers of k and I.

In fact the draw on a three-dimensional plot of $\ell og_{10}(\|e\|) = \varepsilon$ versus I and k shows a uniform convergence of ε to $-\infty$, as k and I grow (see figure 5).

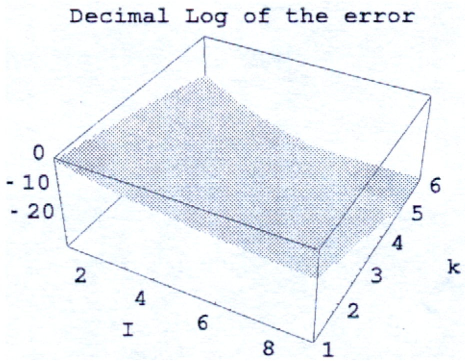


***Figure 5** : Error ε versus k and I*

$$\|e\| = \int_{-\infty}^{+\infty} \left\{ TF\left\{ Sin(x) \right\} - \overline{u}_{k,I}(\theta) \right\}^2 d\,\theta \,.$$

5. Representation of complex systems

The main property of spline functions is to have for coefficients of development of a function the whole partial or complete derivative up to k order of the considered function. This allows to simplify the representation of state or differential equations of as well linear or nonlinear systems by transposing the classical differential and integral calculus in a functional and invariant calculation.

5.1. Linear systems

Let us consider a linear system represented by ;

$$\dot{x} = Ax + Bu$$

where x is the state vector of dimension N , u is the control vector of dimension m, A is the dynamical real matrix of dimension NxN , B is the control real matrix of dimension Nxm.

$x_{k,I}(t)$ the C^k approximation of the state vector $x(t)$ can be written as

$$x_{k,I}(t) = \sum_{\nu=0}^{k} \sum_{i=0}^{I} x_i^{\nu} S_i^{k,\nu}(t)$$

where x_i^{ν} is the ν^{th} derivate of x(t) with respect to t at the node t_i, $S_i^{k,\nu}(t)$ is the ν^{th} C^k Spline function centered at the node t_i and defined on the set $[t_{(i-1)}, t_{(i+1)}]$.

Using the state equation (),the ν^{th} derivate of x(t) can be expressed as,

$$x_i^0 = x_i$$
$$x_i^1 = \dot{x}_i = Ax_i + Bu_i$$
.
.
$$x_i^\nu = A^\nu x_i + \sum_{j=0}^{\nu-1} A^j Bu_i^{(\nu-1-j)}$$

Finally the state vector expression becomes,

$$x_{k,I}(t) = \sum_{i=0}^{I}\sum_{\nu=0}^{k}\left\{A^\nu x_i + \sum_{j=0}^{\nu-1} A^j Bu_i^{(\nu-1-j)}\right\}S_i^{k,\nu}(t)$$

and depends only on x_i i.e. the amplitude of $x(t)$ at the discretized time t_i and of $(\nu-1)$ derivatives of the control vector u(t). and the precision depends on k and ν.

5.2. Differential nonlinear explicit systems

Let us consider a differential nonlinear explicit system defined by,

$$\dot{x} = F(x)$$

Where x is a state vector of dimension N and F is generally an analytic or C^∞ nonlinear function of the state.

We define the vector field $\mathsf{F}[\ ]$ associated to $F[x]$ such as:

$$\mathsf{F}[\ ] = \sum_{i=0}^{N} F^i \frac{\partial}{\partial x_i}$$

We have:

$$\dot{x} = \mathsf{F}[x]$$
$$\ddot{x} = \mathsf{F}[\mathsf{F}[x]] = \mathsf{F}^2[x]$$
.
$$x^\nu = \mathsf{F}^\nu[x]$$

The C^k Spline expansion of x (t) is:

$$x_{k,I}(t) = \sum_{\nu=0}^{k}\sum_{i=0}^{I} x_i^\nu S_i^{k,\nu}(t)$$

Where x_i^ν is the ν^{th} derivate of x (t) with respect to t at the node t_i, $S_i^{k,\nu}(t)$ is the ν^{th} C^k Spline function centered at the node t_i and defined on the set $[t_{(i-1)}, t_{(i+1)}]$.

We obtain:

$$x_{k,I}(t) = \sum_{\nu=0}^{k}\sum_{i=0}^{I} \mathsf{F}^\nu[x_i] S_i^{k,\nu}(t)$$

Which depends only on x_i i.e. the amplitude of x (t) at discretized time t_i.

Example: One-dimensional differential nonlinear system $\dot{x} = x^2$.

$$x_{k,I}(t) = \sum_{\nu=0}^{k}\sum_{i=0}^{I} \nu!(x_i)^{(\nu-1)} S_i^{k,\nu}(t)$$

5.3. Differential nonlinear implicit systems

Let us consider a differential nonlinear implicit system defined by,

$$F(x, \dot{x}, u) = 0$$

where x is the state vector, of dimension N, $\dot{x}$ the derivative of state vector, u the control vector of dimension m and F is an analytic or C^∞ nonlinear function.

Same considerations than in paragraphs 5.1, 5.2 lead to,

$$\sum_{i=0}^{I}\sum_{\nu=0}^{k} \frac{\partial^\nu}{\partial t^\nu} F(x, \dot{x}, u)\Big|_i S_i^{k,\nu}(t) = 0$$

6. Conclusion

We have presented in this paper the main algebraic expressions and properties of the one-dimensional C^k spline functions as well in temporal space as in frequency space. We have also shown that for as well linear as non linear systems , the classical differential and integral calculus can be replaced in C^k splines space by functional and invariant calculation. The major advantages of C^k spline functions can be summarized as follows:

-Coefficients of functional expansion by C^k splines are the same as well in time space as in frequency space. These coefficients are the k derivatives of the considered function, at each point of discretization.

-C^k spines have excellent approximation properties. Interpolation by C^k Spline converges uniformly with excellent rate and precision. We have shown that with K=5 and I=10 the error estimate of sinus spectra approximation is around 10^{-30}. Interpolation by C^k Spline is also available for non uniformly spaced data without any inconvenience.

-C^k splines and their wavelets have excellent localization properties. They open a new way for time-frequency signal analysis.

- The representation of as well linear as non linear complex systems in the space of C^k splines leads to an functional and invariant calculation at place of the classical differential and integral calculus.

The method is to our knowledge the only one which generates a functional space stable through the integral and differential operators and open the way to a new numerical analysis.

This allows to simplify and to rewrite a large number of classical problems, as for example nonlinear optimal control, where the transposition of this differential problem leads to a functional problem in the C^k spline functions spaces, permitting easily the complete elimination of the adjunct vector with the well know associated numerical or differential troubles. Dynamical equations, differential constrains or boundary conditions can be introduced in the C^k spline functional space as algebraic relation transforming the initial space in a dedicated manifolds.

- The Fourier transforms of C^k splines are C^k wavelets with the same good properties opening the way to a new frequency analysis of nonlinear systems.

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