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ROBUST EXPONENTIAL STABILITY OF UNCERTAIN SWITCHED STOCHASTIC SYSTEMS WITH TIME-DELAY

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1 Introduction

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Stochastic dynamic modeling plays an essential role in numerous physics and engineering problems. It can be applied wherever random properties of a dynamical system have to be considered. Most of the emphasis is placed on the stability analysis of the stochastic dynamical systems (see [1, 14, 23]). Moreover, in many applications, the physical or chemical processes are governed by more than one dynamics: the dynamics change among a family of choices with respect to time t or state x. Such processes are often described by switched systems and have been studied extensively in recent years (see [5, 6, 17, 18, 27, 28, 43, 29, 30, 34, 37, 38, 41]). Time-delay and uncertainties are two main causes for instability of dynamical systems (see [12, 21]). Numerous studies have been carried out on stability analysis and stabilization of time-delay systems and uncertain systems (see [3, 4, 7, 8, 9, 11, 13, 15, 20, 24, 26, 25, 31, 33, 35, 39, 40, 42, 44, 45, 46]), some of which have been done in the scope of stochastic systems or switched systems. To the best knowledge of the authors, few work has been done for switched stochastic systems with both uncertainties and time-delay. In addition, the average dwell time scheme associated with the multiple Lyapunov functions has been proved to be an important method for switched systems in [16] (see also [28]). However, these pioneering works have only been conducted for switched systems without delays and stochastic perturbations, although switched systems with delays or stochastic perturbations are in fact very common in practice. Moreover, as long as the stability of a time-delay system is considered, there exist both delay-independent criteria and delay-dependent criteria, of which the latter one is proved to be less conservative and therefore more important in practical applications, although sometimes more challenging to obtain (see [35]). The objective of this paper is to achieve delay-dependent stability analysis of delay switched stochastic systems with uncertainties. The delay-dependent stability criteria are given in terms of linear matrix inequalities (LMIs) and average dwell time.

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2 Problem statement and preliminaries

Consider the following stochastic uncertain switched system

$$\begin{aligned} dx(t) &= [(A_i+\Delta A_i)x(t)+(\tilde{A}_i+\Delta\tilde{A}_i)x(t-h)]dt \\ &\quad +[(B_i+\Delta B_i)x(t)+(\tilde{B}_i+\Delta\tilde{B}_i)x(t-h)]dw(t), \\ &\qquad \text{for} \quad t\geq 0,\ \alpha(t)=i, \qquad (1) \\ x(t) &= \phi(t),\ t\in[-h,0], \end{aligned}$$

where $x\in\mathbb{R}^n$ is the state and h is the constant time-delay. $w(t)$ is a scalar Brownian motion. ϕ is the initial data. $A_i,\tilde{A}_i\in\mathbb{R}^{n\times n}$, B_i and $\tilde{B}_i\in\mathbb{R}^{n\times m}$ are constant real matrices. Let the switching signal $\alpha:\mathbb{R}^+\rightarrow I_K\triangleq\{1,2,...,K\}$ be a piecewise constant and continuous from the right deterministic function, with finitely many discontinuities on every bounded subinterval of $\mathbb{R}^+$. Without loss of generality, we can suppose that $\alpha(t)$ is right continuous at its points of discontinuity. ΔA_i, $\Delta\tilde{A}_i$, ΔB_i and $\Delta\tilde{B}_i$ are time-dependent uncertainties of the form

$$\begin{aligned} \Delta A_i^T\Delta A_i\leq\widehat{A_i}, &\quad \Delta\tilde{A}_i^{\,T}\Delta\tilde{A}_i\leq\widehat{\tilde{A}_i}, \\ \Delta B_i^T\Delta B_i\leq\widehat{B_i}, &\quad \Delta\tilde{B}_i^{\,T}\Delta\tilde{B}_i\leq\widehat{\tilde{B}_i}, \end{aligned}$$

where $\widehat{A_i}$, $\widehat{\tilde{A}_i}$, $\widehat{B_i}$, and $\widehat{\tilde{B}_i}$ are known constant real matrices of appropriate dimensions, which describe the uncertainty bounds. We shall call this kind of uncertainties *admissible uncertainties.* Note that the assumption on the switching signal α guarantees that system (1) switches only finite times on any bounded time interval. It is well-known that, given any initial data $\phi\in L^{2,h}_{\mathcal{F}_0}$, any single system in (1) has a unique solution (see [32]). Based on the assumption on the switching signal, it follows step by step that the switched system (1) has a unique solution as well, denoted by $x(t;\phi)$.

Two types of stability are considered in this paper, one is almost sure exponential stability and the other is exponential stability in pth moment (see [32] for definitions). The main problem now can be formulated as follows:

Problem 2.1. *For all admissible uncertainties, under what conditions will the uncertain switched system* (1) *be almost surely exponentially stable? Under what conditions will it be exponentially stable in mean square?*

3 Main results

In this section, the stability analysis of system (1) is studied.

Theorem 3.1. *Given $h>0$, the uncertain switched stochastic system* (1) *is robustly exponentially stable in mean square if the following two conditions hold:*

1. *For each $i\in I_K$, there exist positive numbers $\varepsilon_{1,i}$, $\varepsilon_{2,i}$, …, $\varepsilon_{9,i}$ and ρ_i, positive definite matrices Q_i, H_i, P_i, and arbitrary matrices G_i such that the following matrix inequalities are satisfied:*

$$\Psi_i=\begin{bmatrix}\Psi_{11} & \Psi_{12} & \Psi_{13}\\ \star & \Psi_{22} & \Psi_{23}\\ \star & \star & \Psi_{33}\end{bmatrix}<0, \qquad (2)$$

$$Q_i\leq\rho_i I, \qquad (3)$$

2. *Let T_a be the average dwell time of system (1),*

$$T_a>\frac{\ln\beta}{\varepsilon}, \qquad (4)$$

where

and ε is the unique root of the equation

$$\lambda-\varepsilon c_2-c_3\varepsilon he^{\varepsilon h}=0. \qquad (5)$$

The following proposition, which can be proved directly by Schur's lemma [2], gives an equivalent version of inequality (2) which can be readily computed with Matlab's LMI control toolbox [10].

Proposition 3.1. *The matrix inequality* (2) *is equivalent to the following inequality:*

$$\Lambda_i=\begin{bmatrix}\Lambda_{11} & \Lambda_{12} & \Lambda_{13} & 0 & 0 & \Lambda_{16}\\ \star & \Lambda_{22} & \Lambda_{23} & \Lambda_{24} & 0 & 0\\ \star & \star & \Lambda_{33} & 0 & \Lambda_{35} & 0\\ \star & \star & \star & \Lambda_{44} & 0 & 0\\ \star & \star & \star & \star & \Lambda_{55} & 0\\ \star & \star & \star & \star & \star & \Lambda_{66}\end{bmatrix}<0, \qquad (6)$$

where

$$\begin{aligned} \Lambda_{11} &= H_i+hP_i+Q_i(A_i+G_i)+(A_i^T+G_i^T)Q_i \\ &\quad +(\varepsilon_{1,i}+h\varepsilon_{3,i})\widehat{A_i}+B_i^TQ_iB_i \\ &\quad +(\rho_i+\varepsilon_{5,i}+\varepsilon_{7,i}+\varepsilon_{8,i}^{-1}\rho_i^2)\widehat{B_i}, \\ \Lambda_{12} &= Q_i(\tilde{A}_i-G_i)+B_i^TQ_i\tilde{B}_i, \\ \Lambda_{22} &= -H_i+(\varepsilon_{2,i}+h\varepsilon_{4,i})\widehat{\tilde{A}_i}+\tilde{B}_i^{\,T}Q_i\tilde{B}_i \\ &\quad +(\rho_i+\varepsilon_{6,i}+\varepsilon_{8,i}+\varepsilon_{9,i})\widehat{\tilde{B}_i}, \\ \Lambda_{13} &= h(A_i^T+G_i^T)Q_iG_i, \\ \Lambda_{23} &= h(\tilde{A}_i^T-G_i^T)Q_iG_i, \\ \Lambda_{33} &= -hP_i, \\ \Lambda_{24} &= \left[\tilde{B}_i^{\,T}Q_i \quad \tilde{B}_i^{\,T}Q_i\right], \\ \Lambda_{44} &= diag\left\{-\varepsilon_{7,i}I,-\varepsilon_{9,i}I\right\}, \\ \Lambda_{35} &= \left[hG_i^TQ_i \quad hG_i^TQ_i\right], \\ \Lambda_{55} &= diag\left\{-\varepsilon_{3,i}hI,-\varepsilon_{4,i}hI\right\}, \\ \Lambda_{16} &= \left[Q_i \quad Q_i \quad B_i^TQ_i \quad B_i^TQ_i\right], \\ \Lambda_{66} &= diag\left\{-\varepsilon_{1,i}I,-\varepsilon_{2,i}I,-\varepsilon_{5,i}I,-\varepsilon_{6,i}I\right\}. \end{aligned}$$

In the previously stated results, we used multiple Lyapunov functionals to deal with switched systems. If a single Q can be found to satisfy inequalities (2) and (3) , we have the following corollary of Theorem 3.1.

Corollary 3.1. *Given $h > 0$, if there exists a positive definite matrix Q such that the matrix inequalities* (2) *and* (3) *are satisfied with $Q_i = Q$, then, for any given switching law, the uncertain switched stochastic system* (1) *is robustly exponentially stable in mean square.*

Remark 3.1. For the almost sure exponential stability, according to [31] or [32], it can be shown that if the conditions of Theorem 3.1 are satisfied, then system (1) is almost surely robust exponentially stable as well. Since under the given conditions almost sure exponential stability can be easily derived from exponential stability in mean square, we shall simply state a system is exponential stability in the following section.

Remark 3.2. To conclude this section, we point out that, for briefness of this paper, we only focus on system with linear uncertainties. However, stability results for system with nonlinear uncertainties can be proved via a similar approach.

4 Application

In this section, we present an example to illustrate our main results. The following example shows that sometimes only multiple Lyapunov functionals would work for the stability analysis.

Example 4.1. Consider the switched system given by

$$\begin{aligned} dx(t) = &\Big[(A_1 + \Delta A_1)x(t) + (\tilde{A}_1 + \Delta\tilde{A}_1)x(t-h)\Big]\,dt \\ &+ g_1(t, x(t), x(t-h))dw(t), \quad \alpha(t) = 1, \end{aligned} \tag{7}$$

and

$$\begin{aligned} dx(t) = &\Big[(A_2 + \Delta A_2)x(t) + (\tilde{A}_2 + \Delta\tilde{A}_2)x(t-h)\Big]\,dt \\ &+ g_2(t, x(t), x(t-h))dw(t), \quad \alpha(t) = 2, \end{aligned} \tag{8}$$

where

$$A_1 = \begin{bmatrix} -1.9 & -3.3 \\ 7.6 & -11.3 \end{bmatrix}, \quad \tilde{A}_1 = \begin{bmatrix} -4 & 8.9 \\ 1.8 & -2.5 \end{bmatrix},$$

$$A_2 = \begin{bmatrix} -5.4 & 3.9 \\ 0 & -7.2 \end{bmatrix}, \quad \tilde{A}_2 = \begin{bmatrix} 5 & 2.6 \\ -2.2 & -2.1 \end{bmatrix},$$

$$\Delta A_1^T \Delta A_1 \leq \widehat{A}_1 = 0.01I, \quad \Delta\tilde{A}_1^T \Delta\tilde{A}_1 \leq \widehat{\tilde{A}}_1 = 0.01I,$$

$$\Delta A_2^T \Delta A_2 \leq \widehat{A}_2 = 0.01I, \quad \Delta\tilde{A}_2^T \Delta\tilde{A}_2 \leq \widehat{\tilde{A}}_2 = 0.01I,$$

and

$$\begin{aligned} &\mathrm{tr}\left[g_1^T(t, x(t), x(t-h))g_1(t, x(t), x(t-h))\right] \\ &\quad \leq 0.1\,\|x(t)\|^2 + 0.1\,\|x(t-h)\|^2, \\ &\mathrm{tr}\left[g_2^T(t, x(t), x(t-h))g_2(t, x(t), x(t-h))\right] \\ &\quad \leq 0.1\,\|x(t)\|^2 + 0.1\,\|x(t-h)\|^2. \end{aligned}$$

Table 1: Stability bounds of time-delay and average dwell time (Example 4.1)

T_0 ($\times 10^3$)	0	0.0856	0.2303	0.7960
h_0	0	0.5	1.0	1.5
T_0 ($\times 10^3$)	1.7645	3.0717	5.1803	9.2734
h_0	2.0	2.5	3.0	3.5
T_0 ($\times 10^3$)	23.4052	238.2837		
h_0	4.0	4.4		

It can be shown using Matlab's LMI control toolbox that there exists no single Q to suit the inequalities (2) and (3). However, if we apply Theorem 3.1 instead of Corollary 3.1, we can see that the switched system given by systems (7) and (8) is robustly exponentially stable for $0 \leq h \leq 4.4$. For different average dwell time lower bounds T_0, the delay upper bounds h_0 guaranteeing the exponential stability of the system are listed in Table 1.

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