MINORITY GAME: MESOSCOPIC EXPLANATION OF MACROSCOPIC OBSERVABLES IN HERD REGIME

Karol Wawrzyniak and Wojciech Wiślicki *

Abstract. We present the study of utility function of the minority game in the efficient regime for two different payoff functions. We develop an effective mesoscale description of the state, represent the game as the Markov process and prove finiteness of the number of states. Using Markov process theory we can explain all interesting observable features of the macroscopic observables like behavior of variance per capita and predictability of the aggregated demand. We prove that in the case of linear payoff many attractors in the state space are possible.

Keywords. Minority game; adaptive system; Markov process; mesoscopic model;

1 The Introduction

Minority game (MG) was designed [1] as a microscopic model of adaptive behaviour observed in multi-agent systems. The MG is a typical bottom-up construct and therefore usual definitions of the game first specify rules of behaviour for individuals. Then, piecing together microscopic variables, one defines higher-order quantities characterizing grander systems. In some cases, however, other constructs are also possible, e.g. functions of state like score functions can be attributed to groups of agents without specifying agents individually (cf. ref. [2]). Despite simplicity of basic rules of taking decisions by agents, adaptive abilities and phenomenology of populations playing MGs appear to be surprisingly interesting and their properties are non-trivial [3]. Special studies were devoted to understanding such functions like aggregated demand, market volatility, market occupancy etc. It was shown [4][5] that the MG exhibits different modes of behaviour, depending on the game parameters: the random, cooperation and herd. The latter case is characterized by small strategy space compared to the overall number of agents. Following authors of ref. [6] we prefer to call this regime efficient, because all players have all available information at their disposal. Our study of this regime is motivated by interesting phenomenology observed in numerical simulations. Starting from macroscopic phenomenology we show that the most observations can be explained only if going to underneath layers i.e. mesoscopic and microscopic layers that are defined as follows.

In physics, the macroscopic world contains things one can see with its eyes. In the MG the only public observable is the aggregate demand $A(t)$. By that means all the analysis that is based on $A(t)$ and does not look deeper into underlying mechanism can be classified as macroscopic.

The microscopic world contains the building blocks of matter: the atoms and molecules that are not observed directly. In the MG this level is related to examining the time evolution of each strategy separately. Knowing it also the agents' actions can be easily derived. At this level we analyze quantities unavailable for individual agents.

Between the microscopic and macroscopic world is the mesoscopic one. The boundaries are not sharp, but can be roughly indicated. Besides quantitative differences between laws of nature, on the mesoscopic level sets of particles with similar properties are treated as a one object. Analogically in MG the group of strategies that behave identically may be considered as a whole. Much as in physics, where the detailed understanding of the microscopic and mesoscopic world provides invaluable insight on macroscopic phenomena, we believe that a consistent picture of the microstructure mechanisms will help put in perspective some of the traditional questions about macroscopic behavior in MG such as: 'Why does the aggregated demand exhibit large-amplitude oscillations?'[5] Why is the demand periodic in time?'[6][7][5][8] or 'How the dynamics is influenced by the payoff function?'. Dealing with above questions has been recently put forth. The crowd-anticrowd theory [9][10] presents acceptable explanation for oscillations but fails to deal with the periodicity. This issue was treated by the authors of ref. [7] and, more fruitfully, ref. [2]. The authors of ref. [2] introduced the concept of the state of the MG but limit their analysis to the reduced strategy space. Different approach is presented in book [11] where the author incorporates the theory of generating functionals. This theory is tough mathematically and does not provide with explanation for all observed phenomena. Hence, in our recent paper [12] we furnish a novel concept that allows to convert the MG to the Markov Process and provides answers for above questions. Nevertheless our considerations presented there are mostly limited to the payoff

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$g(x) = \mathrm{sgn}(x)$. In this paper the quantitative model on mesoscale is extended to two particular payoffs the step-like and linear. The reasoning used for the latter one may be extended to other types. In addition, we analyze how disproportions between groups of agents possessing different kind of strategies influence the game. Hence, the presented reasoning provides a consistent explanation for all observables in the herd regime.

2 The Formal Definition of the Minority Game

At each time step t, the nth agent out of N ($n = 1, \ldots, N$) takes an action $a_{\alpha_n}(t)$ according to some strategy $\alpha_n(t)$. The action $a_{\alpha_n}(t)$ takes either of two values: -1 or $+1$. An aggregated demand is defined

$$A(t) = \sum_{n=1}^{N} a_{\alpha'_n}(t), \tag{1}$$

where α'_n refers to the action according to the best strategy, as defined in eq. (3) below. Such defined $A(t)$ is the difference between numbers of agents who choose the $+1$ and -1 actions. Agents do not know each other's actions but $A(t)$ is known to all agents. The minority action $a^*(t)$ is determined from $A(t)$

$$a^*(t) = -\mathrm{sgn}A(t). \tag{2}$$

Each agent's memory is limited to m most recent winning, i.e. minority, decisions. Each agent has the same number $S \geq 2$ of devices, called strategies, used to predict the next minority action $a^*(t+1)$. The sth strategy of the nth agent, α_n^s ($s = 1, \ldots, S$), is a function mapping the sequence μ of the last m winning decisions to this agent's action $a_{\alpha_n^s}$. Since there is $P = 2^m$ possible realizations of μ, there is 2^P possible strategies. At the beginning of the game each agent randomly draws S strategies, according to a given distribution function $\rho(n) : n \to \Delta_n$, where Δ_n is a set consisting of S strategies for the nth agent.

Each strategy α_n^s, belonging to any of sets Δ_n, is given a real-valued function $U_{\alpha_n^s}$ which quantifies the utility of the strategy: the more preferable strategy, the higher utility it has. Strategies with higher utilities are more likely chosen by agents.

There are various choice policies. In the popular *greedy policy* each agent selects the strategy of the highest utility

$$\alpha'_n(t) = \arg \max_{s:\, \alpha_n^s \in \Delta_n} U_{\alpha_n^s}(t). \tag{3}$$

If there are two or more strategies with the highest utility then one of them is chosen randomly. The highest-utility strategy (3) used by the agent is called the *active strategy*, in contrast to *passive strategies*, unused at given moment. However, at any time all agents evaluate all their strategies, the active and passive ones. Each strategy α_n^s is given the *payoff* depending on its action $a_{\alpha_n^s}$

$$R_{\alpha_n^s}(t) = -a_{\alpha_n^s}(t)\, g[A(t)], \tag{4}$$

where g is an odd *payoff function*, e.g. the steplike $g(x) = \mathrm{sgn}(x)$ [4], proportional $g(x) = x$ or scaled proportional $g(x) = x/N$. The learning process corresponds to updating the utility for each strategy

$$U_{\alpha_n^s}(t+1) = U_{\alpha_n^s}(t) + R_{\alpha_n^s}(t), \tag{5}$$

such that every agent knows how good its strategies are.

3 The Macroscopic perspective

Up to now the most spectacular result is related to behavior of variance per capita $\sigma(A)^2/N$ as a function of the control parameter $N/2^m$ [4][5], where

$$\sigma^2(A) = \frac{1}{T} \sum_{t=0}^{T} A(t)^2 \tag{6}$$

In Figs 1 this quantity is displayed for two MGs differing by the memory length, for the steplike (full line) and proportional (dashed line) payoffs. It is seen that two payoffs give results very close to each other what, at first sight, could indicate that the game is insensitive to the payoff form [13]. Similar premise is given by another macroscopic observable i.e. the predictability H_a. The H_a was introduced in ref. [14] and is defined as:

$$H_a = \frac{1}{P} \sum_{\mu=1}^{P} \langle a^* | \mu \rangle^2 \tag{7}$$

where $\langle a^* | \mu \rangle$ is the conditional average of a^* given μ, and H_a is its mean-squared value over all P histories. In literature e.g. [14][3] the analyzed variable is divided by N. We suspect that this division was dictated by willingness to consistently observe all macroscopic variables as *per capita*. However, we show further that such approach can lead to some flaws in H_a analysis if not performed carefully. The H_a was demonstrated to be useful in detecting two interesting phases of the MG:

- The *symmetric phase* with $H_a \simeq 0$, where after the particular history $\mu(t)$ both signs of $a^*(t)$ appear with the same frequency. In lit. [15][14] is stated manifold that if $H_a = 0$ then patterns in the signal do not exist. We find this condition to be necessary but not a sufficient one to state the lack of patterns. For example, if every appearance of given μ is followed by negative and positive minority decision alternately then $H_a = 0$ and the predictable pattern exists. Indeed, such a behaviour is observed for the MG in the herd regime and for $g(x) = x$ [12]. Hence, H_a measures disproportions in frequencies between positive and negative minority decisions rather than detects patterns.

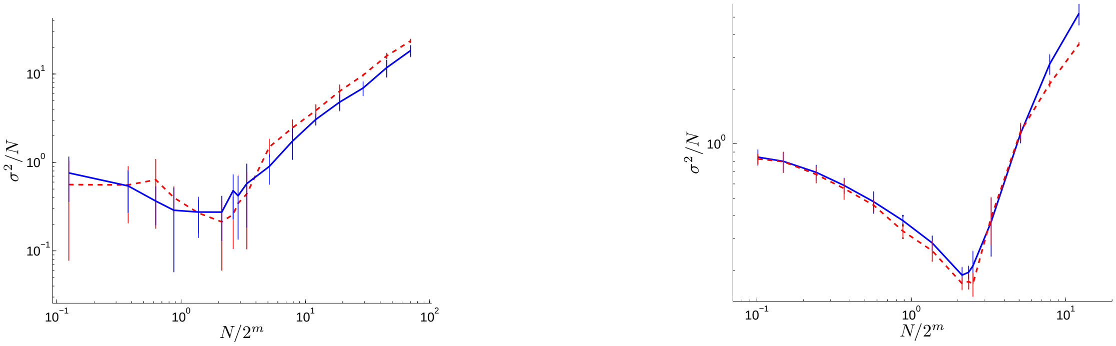


Figure 1: σ^2/N as a function of $n = N/P$ for s = 2, m = 3 (left) or $m = 7$ (right). Two different payoff functions are used i.e. full lines correspond to $g(x) = \text{sgn}(x)$ and dashed lines to $g(x) = x$. Each point is a mean from ten games, error bars correspond to one standard deviation and curves are drawn to guide ones eye.

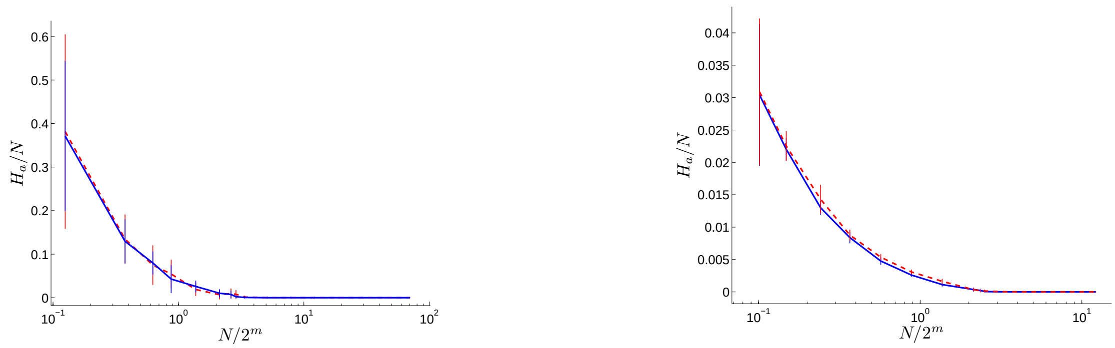


Figure 2: H_a/N as a function of $n = N/P$ for s = 2, m = 3 (left) or $m = 7$ (right). Two different payoff functions are used, i.e. full lines correspond to $g(x) = \text{sgn}(x)$ and dashed lines to $g(x) = x$. Each point is a mean from ten games, error bars correspond to one standard deviation and curves are drawn to guide ones eye.

- The *asymmetric phase* with $H_a > 0$ and existing predictable patterns. In the asymmetric phase, sign predictions significantly better than random are possible.

As presented in Fig. 2 plots of H_a/N drown for different payoffs are very close to each other. By that means in the early literature it was falsely conjectured that only the payoff's evenness is relevant to the macroscopic observables [13]. Nevertheless, the failure of this conclusion is easily visible if slightly modified macroscopic observable H_A is introduced.

$$H_A = \frac{1}{P}\sum_{\mu=1}^{P}\langle A|\mu\rangle^2 \tag{8}$$

The plot of H_A/N (cf Fig. 3) shows some sensitivity of this quantity to the payoff form. In order to investigate this relation, first we decompose $\langle a^*|\mu\rangle$ into two following ingredients $\langle a^*|\mu\rangle = \langle a^*_-|\mu\rangle + \langle a^*_+|\mu\rangle$, where

$$\langle a^*_+|\mu\rangle = \frac{1}{T}\sum_{t=1}^{T} a^*(t)\delta\big(\mu(t),\mu\big)\delta\big(a^*(t),1\big) \tag{9}$$

$$\langle a^*_-|\mu\rangle = \frac{1}{T}\sum_{t=1}^{T} a^*(t)\delta\big(\mu(t),\mu\big)\delta\big(a^*(t),-1\big) \tag{10}$$

,where $\delta(x, y)$ stands for the Kronecker symbol. Similarly, decomposition of $\langle A|\mu\rangle$ (Eq. 8) is as follows: $\langle A|\mu\rangle = \langle A^-|\mu\rangle + \langle A^+|\mu\rangle$ where,

$$\langle A_+|\mu\rangle = \frac{1}{T}\sum_{t=1}^{T} A(t)\delta(\mu(t),\mu)\delta(sgnA(t),1) \tag{11}$$

$$\langle A_-|\mu\rangle = \frac{1}{T}\sum_{t=1}^{T} A(t)\delta(\mu(t),\mu)\delta(sgnA(t),-1) \tag{12}$$

The $H_a = 0$ is only possible if for every μ: $\langle a^*|\mu\rangle = 0$ what subsequently requires that $|\langle a^*_-|\mu\rangle| = |\langle a^*_+|\mu\rangle|$. This means that positive and negative values of $A(t)$ have to appear with the same frequency. Similarly, the $H_A = 0$ is only possible if for every μ: $|\langle A_-|\mu\rangle| = |\langle A_+|\mu\rangle|$ i.e. the negative and positive values compensate mutually.

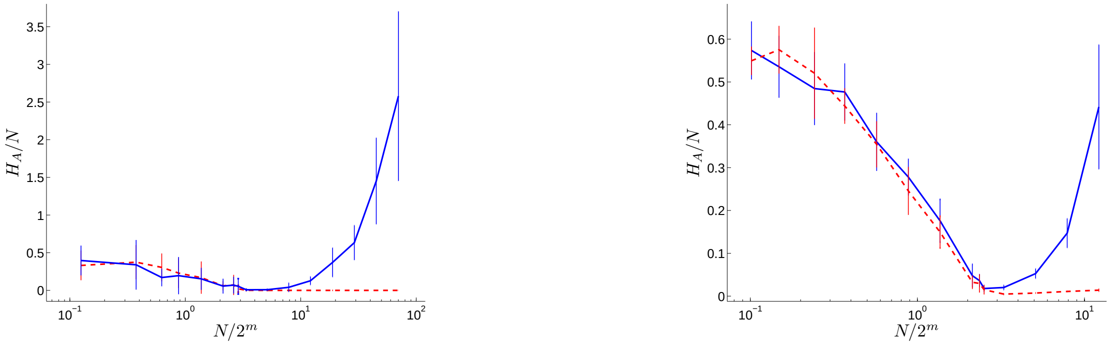


Figure 3: H_A/N as a function of $n = N/P$ for s = 2, m = 3 (left) or $m = 7$ (right). Two different payoff functions are used, i.e. full lines correspond to $g(x) = \text{sgn}(x)$ and dashed lines to $g(x) = x$. Each point is a mean from ten games, error bars correspond to one standard deviation and curves are drawn to guide ones eye.

Combinations like (i) $H_a = 0$ and $H_A > 0$, (ii) $H_a > 0$ and $H_A = 0$ are also possible. Further explanation why the H behavior is sensitive to the payoff requires deeper analysis on the mesoscopic perspective.

4 The Mesoscopic perspective

4.1 The Fraction - definition and statistical properties

The fraction can be defined in a two ways. In a first approach it is a set of agents that possess a given all the same set of S strategies. The number of agents in a fraction (the size of a fraction) is marked as F_ν where $\nu = \{1 \ldots G\}$ and G is the number of different fractions. In large games, the system comprises agents of all possible fractions what results in constant G. Considering F_ν, if strategies are assigned to agents randomly then F_ν is a random variable. As the strategy space consists of 2^P possible strategies, G is represented by the number of combinations with repetition: $G = \binom{2^P+S-1}{S}$. However, such definition of G makes the expected values of sizes $E[F_\nu]$ not equal for different fractions provided that strategies are drawn from the uniform distribution. For example, assuming $S = 2$, the fraction with two the same strategies, .e.g. $\{\beta_1, \beta_1\}$, is two times smaller than fraction with strategies β_1 and β_2: $\{\beta_1, \beta_2\}$ or $\{\beta_2, \beta_1\}$. Hence, we further use another definition of fraction i.e. the fraction is as a set of agents characterized by a given sequence of strategies, i.e. $G = 2^{PS}$. Such approach is not strict from the MG point of view as the index $S \in \{1, \ldots, S\}$ does not play any role for an agent. We use it because it radically simplifies further analysis and does not bias the outcome, assuming assigning agents to fractions with equal probability. If at the beginning of the game strategies are drawn with equal probability, it corresponds with assigning agents to a specific fraction with probability $\frac{1}{G}$. If the W_ν^n is a random variable $W_\nu^n = \{0, 1\}$ and is equal 1 if agent n belongs to fraction ν then $F_\nu = \sum_{n=1}^{N} W_\nu^n$ follows the binomial distribution $Pr(F_\nu = f_\nu) = \binom{N}{f_\nu}(\frac{1}{G})^{f_\nu}(1 - \frac{1}{G})^{N-f_\nu}$. Hence, $E[F_\nu] = N/G$ or, if normalized, $E[F_\nu/N] = 1/G$.

For $N \to \infty$ we have $Var[F_\nu] \to \infty$ and[1] $Var[F_\nu/N] \to 0$, which means that in the limit (i) the absolute differences between sizes of fractions grow (ii) percentages of population assigned to any fraction are equal. Hence, the larger the population, the larger the expected difference between a real size of fraction F_ν and the expected value $E[F_\nu]$.

4.2 The stability for large games

The game is considered stable if for each strategy $\langle \Delta U \rangle = 0$, i.e. the $U(t)$ has to be mean reverting process. The MG has a build-in stabilization mechanism provided that the game is large enough. The explanation is as follows. Imagine that a certain group of $Z \subset \{\beta_1, \ldots, \beta_{2^P}\}$ strategies get on average a higher payoff and hence their U grows. Then, there always exist the same number of anticorrelated strategies that have decreasing utility. The probability that an agent uses one of the strategies with a high utility is $1 - (\#Z/2^P)^S$, compared to those who use strategies with a low utility $(\#Z/2^P)^S$ [12]. Because the former probability is always higher, provided that $S \geq 2$, then most of population uses better strategies what causes that their utility decreases (is stabilized). As long as proportions between fractions are close to each other, the above mechanism works and game stays stable.

4.3 The payoff $g(x) = \text{sgn}(x)$

We showed in ref. [12] that the game with the step-like payoff can be represented as a Markov Process (MP) where a state is described using $\mu(t)$ and a set of utilities

[1] After the normalization the Bernoulli dist. is $Z_\nu^n = W_\nu^n/N = \{0, \frac{1}{N}\}$ with $P(Z_\nu^n = 0) = 1 - \frac{1}{G}$ and $P(Z_\nu^n = \frac{1}{N}) = \frac{1}{G}$. Hence, $E[Z_\nu^n] = \frac{1}{GN}$ and $Var[Z_\nu^n] = \frac{1}{GN^2}(1 - \frac{1}{G})$. As a result $Var[F_\nu/N = \sum_N Z_\nu^n] = \frac{1}{GN}(1 - \frac{1}{G})$

for the complete set of 2^P pairwise different[2] strategies $\{\beta_\iota\}_{\iota=1}^{2^P}$:

$$x(t) = [\,\mu(t),\, U_{\beta_1}(t), U_{\beta_2}(t), \ldots, U_{\beta_{2^P}}(t)\,]. \tag{13}$$

Nevertheless the reasoning presented in ref. [12] is strictly true only in the ideal case where subpopulations of agents in different fractions are equal or if the system is considered *a priori* i.e. before strategies are assigned to agents at the beginning of the game. In *a posteriori* analysis we take into account a game where strategies are already assigned. In most cases such game is characterized by an inequality between fractions due to the initial randomness in the strategies' generation process. We show that some interesting phenomena, among them the H sensitivity for the payoff, appear only when the disproportion between fractions exists.

4.3.1 The Markov Process representation

The MG can be described in terms of the Markov process with the finite number of states. Generally, the $\mathrm{sgn}(A(x_i))$ takes only two values and hence two successive states are possible. Nevertheless, in some specific states the $A(x_i)$ is always positive or negative and only one value of $\mathrm{sgn}(A(x_i))$ appears. Hence, the transition may be either stochastic or deterministic and the transition probability is equal to

$$Pr(x_j|x_i) = \frac{1}{2}(E[\mathrm{sgn}(A(x_i))] + 1). \tag{14}$$

Using the concept of fractions, we redefine the $A(x_i)$ as follows:

$$A(x_i) = \sum_\nu C_\nu(x_i) F_\nu, \tag{15}$$

where $C_\nu(x_i) \in [-1, 1]$ is the action of the fraction j in the state x_i,

$$C_\nu(x_i) = \frac{1}{F_\nu} \sum_{n=1}^{F_\nu} a_{\alpha'_n(x_i)} \tag{16}$$

The $C_\nu(x_i)$ depends on the action suggested in x_i by the best strategy/ies of this fraction. There are the following groups of fractions:

- A fraction with only one best strategy in x_i. All agents assigned to the fraction react according to the action suggested by this strategy.
- A fraction with many best strategies suggesting the same action in x_i. Although agents use different strategies they all react identically.
- A fraction with many best strategies suggesting opposite actions in x_i. Agents decisions are possibly inhomogeneous and the action of such fraction is a random variable $C_\nu(x_i)$ that takes values $c_\nu^\varphi = \frac{-F_\nu + 2\varphi}{F_\nu}$ for $\varphi = \{0, \ldots, F_\nu\}$. The distribution over these values depends on a proportion between best strategies suggesting opposite actions. Assuming there is $p^{+(-)}$ strategies suggesting the positive (negative) action the $C_\nu(x_i)$ follows the binomial distribution

$$Pr(C_\nu(x_i) = c_\nu^\varphi) = \binom{F_\nu}{\varphi} \Big(\frac{p^+}{p^+ + p^-}\Big)^\varphi \Big(\frac{p^-}{p^+ + p^-}\Big)^{F_\nu - \varphi}. \tag{17}$$

Fractions from the first two groups and suggesting $+1$ are marked with o, suggesting -1 are marked with q, and those belonging to the third group are indexed with r. Hence, Eq. 15 transforms to:

$$\begin{aligned} A(x_i) \;\; &= \sum_{F_o : C_o(x_i) = 1} F_o - \sum_{F_q : C_q(x_i) = -1} F_q \\ &+ \sum_{F_r : C_r(x_i) \in [-1, +1]} C_r(x_i) F_r. \end{aligned} \tag{18}$$

Further analysis is relatively easy when fractions are equal and complicates if random.

The case of equal fractions: There are the same numbers of agents per fraction. We further call such system and the corresponding MP process as the *reference system* and *reference MP*. The $A(x_i)$ is a random variable:

$$A(x_i) = \frac{N}{G}\big(O - Q + \sum_{F_r : C_r(x_i) \in [-1, +1]} C_r(x_i)\big) \tag{19}$$

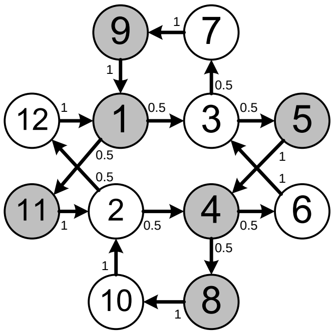
where O and Q refer to the total numbers of fractions from the first two groups suggesting $+1$ and -1, respectively. If the state is deterministic then the components with opposite signs do not compensate and

$$|O - Q| > \max\big(\sum_{F_r : C_r(x_i) \in [-1, +1]} C_r(x_i)\big). \tag{20}$$

Eq. (20) is true always if the negative and positive components are unbalanced, i.e. $O \neq Q$. This can be proved at least in two ways. The general proof is introduced in ref. [12] and uses ordered lists of utilities. Another, more complicated approach is based on fractions. It requires separate analyses per state as given in Appendix A. For stochastic transitions always $O = Q$. For such states, $A(x_i)$ has distribution symmetric around zero which indicates that also the distribution of $\mathrm{sgn}(A(x_i))$ is symmetric and $E[\mathrm{sgn}(A(x_i))] = 0$. The transitions to two following states are equally probable. Knowing how to distinguish the deterministic and stochastic states the algorithm of defining the MP is as follows:

1. Consider separately all 2^m initial states. Such states are characterized by $U_{\beta_\iota} = 0$ for all ι and different μ-s. Due to equality of all strategies the two minority decisions are equally possible for each of initial states and a transition is stochastic. These minority decisions determine strategies that get positive/negative payoff. The updated U and μ values determine 2^{m+1} next states.

[2] Two strategies are called different if their Hamming distance is not equal to zero. The number of pairwise different strategies is equal to 2^P.



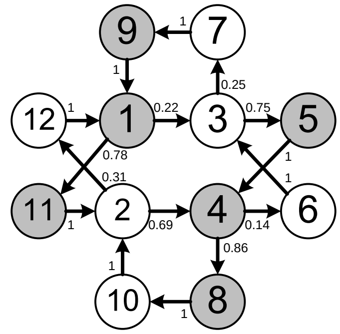


Figure 4: Diagrams of the Markov chain representation of the MG in the efficient regime for $m = 1$. Numbers in circles represent the following states: $x_1 = [-1, 0, 0, 0, 0]$, $x_2 = [1, 0, 0, 0, 0]$, $x_3 = [1, -1, -1, 1, 1]$, $x_4 = [-1, 1, -1, 1, -1]$, $x_5 = [-1, 0, -2, 2, 0]$, $x_6 = [1, 0, -2, 2, 0]$, $x_7 = [1, -2, 0, 0, 2]$, $x_8 = [-1, 2, 0, 0, -2]$, $x_9 = [-1, -1, -1, 1, 1]$, $x_{10} = [1, 1, -1, 1, -1]$, $x_{11} = [-1, 1, 1, -1, -1]$, $x_{12} = [1, -1, 1, -1, 1]$. Values assigned to arrows reflect transition probabilities. Two cases are shown: equal fractions (left) and unequal ones where agents draw strategies with uniform probability (right). States marked as gray incorporate $\mu = -1$ while the white ones $\mu = +1$ (for details see [12]).

2. If in the next state there are many best pairwise different strategies suggesting opposite actions then $O = Q$ and, again, two minority decisions and two successive states are possible, and the transition is stochastic. Hence, two next states have to be determined.

3. If in the next state there are many best pairwise different strategies suggesting the same action then $O \neq Q$ the minority decision is determined by this action and transition is deterministic.

4. If there is only one strategy characterized by the highest value of the utility then $O \neq Q$ and the minority decision is determined by the best strategy and transition is deterministic.

An example realization of the $A(t)$ for the reference MP is given in Fig. 5 (upper left). The estimated A distribution is symmetric (upper right) and the distribution of $\mathrm{sgn}(A)$ is symmetric likewise (lower right). The scatter plot of $A(t+\tau)$ as a function of $A(t)$, where $\tau = 2^{m+1}$ indicates periodicity and existence of preferred values of A (lower left). The reconstructed MP is shown in Fig. 4 (left). Using the MP representation the behavior of $A(t)$ described above is easily understood [12].

The case of unequal fractions: Any case where strategies are assigned randomly to agents leads most likely to the inequality in fractions' sizes. We further consider one of the simplest cases where strategies are drawn with equal probability which corresponds to assigning an agent to any fraction with the probability $\frac{1}{G}$. Interestingly, numerical experiments show that in most cases the reconstructed MP follows the sequence of states of the reference MP but the values of transition probabilities are not reproduced. This bias does not disappear even if the game is enlarged (see Figs 4 and 6). The explanation is as follows.

States in the reference MP where stochastic transition appears are characterized by the same number of positive and negative components in formula (18). Calculating the transition probability we found it crucial to distinguish two scenarios before and after assignment of strategies to agents. Hence we distinguish between *a priori* and *a posteriori* aggregate demands. We denote *a priori* demand by $A(x_i)$ and *a posteriori* by $\widetilde{A}(x_i)$. Calculating *a priori* value we do not know yet the specific number of agents in the νth fraction and we just operate on random variables:

$$\begin{aligned} E[\mathrm{sgn}(A(x_i))] &= E\Big[\mathrm{sgn}(\sum_{\nu} C_\nu(x_i) F_\nu\Big] && (21) \\ &= E\Big[\mathrm{sgn}\big(\sum_{F_o : C_o(x_i)=1} F_o - \sum_{F_q : C_q(x_i)=-1} F_q \\ &\quad + \sum_{F_r : C_r(x_i)\in[-1,+1]} C_r(x_i) F_r V\big)\Big] && (22) \end{aligned}$$

Each F has the same binomial distribution. Since we consider stochastic transitions in the reference system then there is the same number of elements in first and second sum. The distribution of the third sum is symmetric around zero because it contains pairwise symmetric components. Thus, the distribution of $A(x_i)$ is also symmetric as well as the distribution of $\mathrm{sgn}(A(x_i))$. By that means $E[\mathrm{sgn}(A(x_i))] = 0$. When the strategies are assigned to agents then the number of agents in fractions, f_ν, are known. The $E[\mathrm{sgn}(\widetilde{A}(x_i))]$ decomposes as follows:

$$\begin{aligned} E[\mathrm{sgn}(\widetilde{A}(x_i))] &= E\Big[\mathrm{sgn}\big(\sum_{f_o : C_o(x_i)=1} f_o - \sum_{f_q : C_q(x_i)=-1} f_q \\ &\quad + \sum_{f_r : C_r(x_i)\in[-1,+1]} C_r(x_i) f_r\big)\Big]. && (23) \end{aligned}$$

Provided $S = 2$, the last ingredient of the sum (23) is symmetric around zero due to C^r symmetry but the first

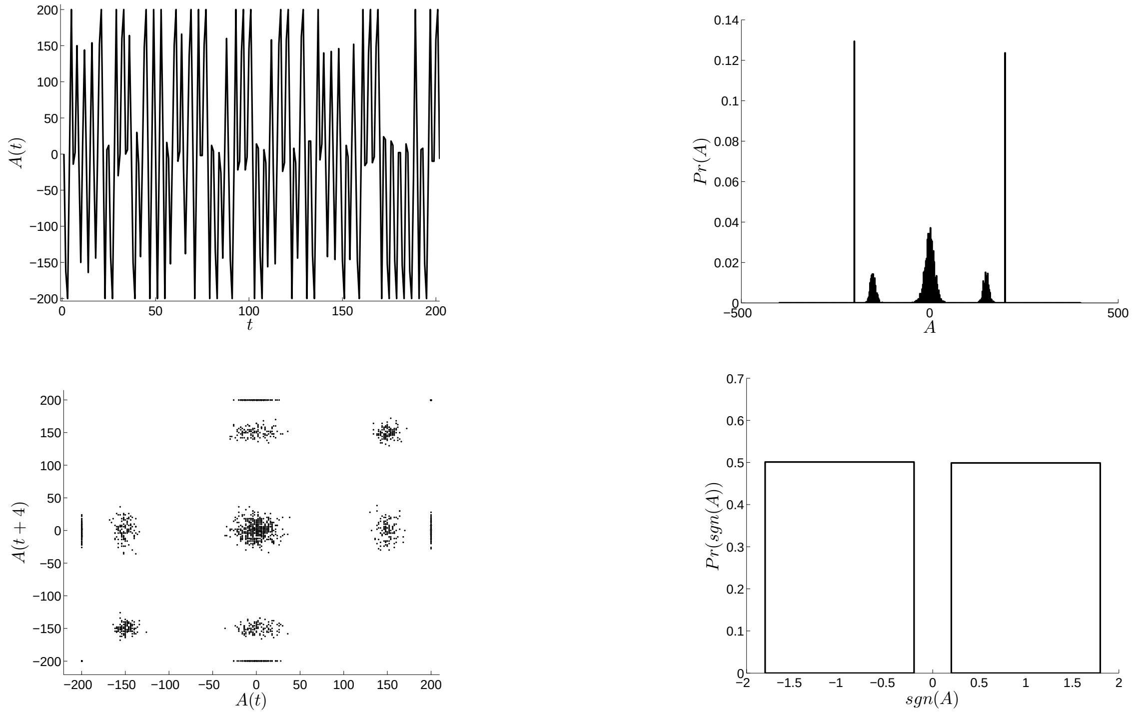


Figure 5: Time evolution of the aggregated demand $A(t)$ (upper left), Plots of the aggregated demand $A(t+2\cdot 2^m)$ vs. $A(t)$ (lower left), Estimated $Pr(A)$ (upper right) and $Pr(\mathrm{sgn}(A))$ (lower right) for the population size $N=400$ and agent memory $m=1$, $S=2$ strategies per agent and identical sizes of fractions

two ingredients introduce a bias and shift the distribution of $A(x_i)$. If $S>2$ then also the third ingredient may be biased. We know that if $N\to\infty$ then $Var[F]\to\infty$. Hence, considering only two components,

$$Var[\sum_{F_o:C^o(x_i)=1} F_o - \sum_{F_q:C_q(x_i)=-1} F_q]\to\infty \qquad (24)$$

what means that the probability of appearance of a large bias grows with N. If the distribution of $A(x_i)$ is shifted then also the distribution of $\mathrm{sgn}(A(x_i))$ is asymmetric regardless of a symmetry of the last ingredient. As a result, most likely, $E[\mathrm{sgn}(\widetilde{A}(x_i))]\neq 0$. The equality $E[\mathrm{sgn}(\widetilde{A}(x_i))] = E[\mathrm{sgn}(A(x_i))] = 0$ is true only when numbers of agents per fraction are equal for all fractions. In other cases the expected absolute bias of distribution increases with N, stochastic probabilities are most likely unequal and the distribution of $\widetilde{A}$ is most likely asymmetric either. In some experiments we found that for some states the bias can shift the distribution so heavily that the distribution is always positive or negative. Thus, the state that is stochastic when analyzed *a priori* may become deterministic when analyzed *a posteriori*.

Finally, consider the states with deterministic transitions in the reference system. If now the F is a random variable then with some (usually very small) probability the transition becomes a stochastic due to a specific realization of F. The analysis of one exemplified state is given in the appendix.

4.3.2 Stability and H behavior

Disproportions in fractions affect transition probabilities. If the absolute disproportions are very large then some transitions that exist in the reference system can disappear and the graph is reduced to its subgraph. The game remains stable because every subgraph is characterized by sequence of states that assures that $+1$ and -1 appear after given μ with the same frequency (white and gray circles respectively in Fig 4). The same frequency of opposite minority decisions after any μ is a necessary and sufficient condition to assure stability provided $g(x)=\mathrm{sgn}(x)$. Hence, the stability forces the same frequency what leads to $\langle a^*|\mu\rangle = 0$. Thereby no matter if the system is the reference one or not the H_a is always equal to zero, provided the game is stable. Above works as long as the game is deep in herd regime, i.e. $NS\gg 2^P$, and if strategies are drawn from the uniform distribution or one close to it. If game moves to the cooperation mode or strategies are drawn from an asymmetric distribution then the methodology of MP breaks because relative disproportions between fractions are large what distorts the stability and additional states start to appear.

The stability mechanism requires balance between the frequency of negative and positive signs of A after any μ, regardless of the value of A. The $\langle A|\mu\rangle$ in formula (8) can

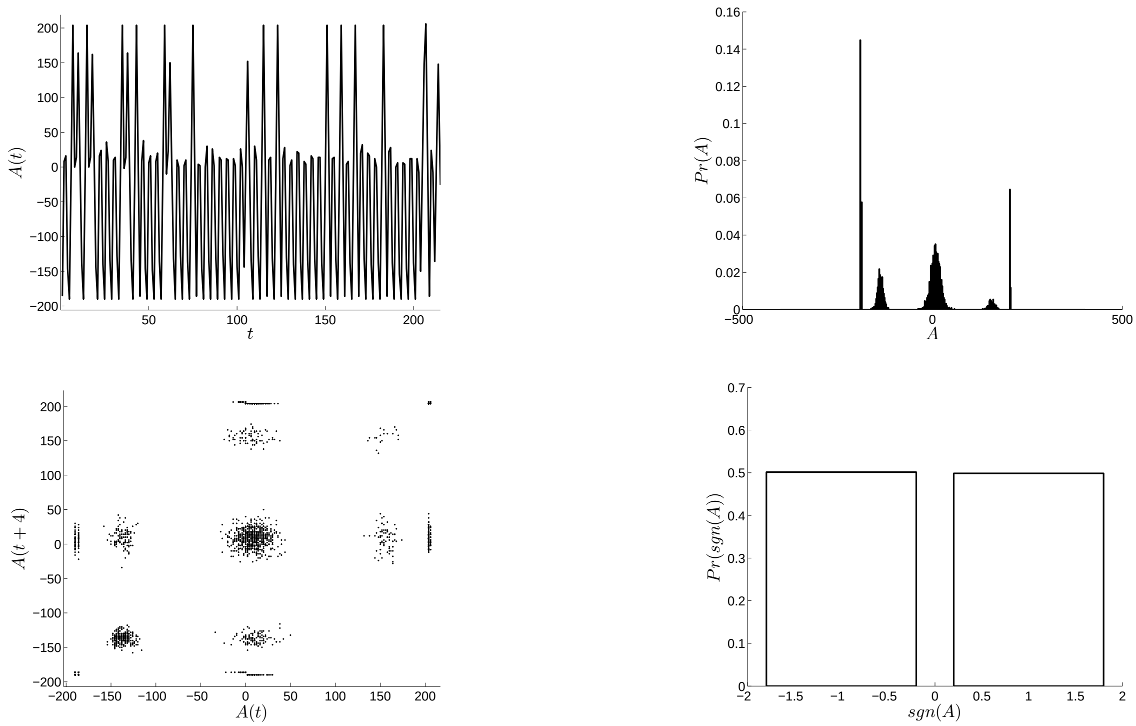


Figure 6: Time evolution of the aggregated demand $A(t)$ (upper left), Plots of the aggregated demand $A(t+2\cdot 2^m)$ vs. $A(t)$ (lower left), Estimated $Pr(\widetilde{A})$ (upper right) and $Pr(\mathrm{sgn}(\widetilde{A}))$ (lower right) for the population size $N=400$ and agent memory $m=1$, $S=2$ strategies per agent and unequal sizes of fractions

be redefined as follows:

$$\langle A|\mu\rangle \simeq \sum_{X^\mu} E\big[A(x_i)\big]Pr(x_i) \tag{25}$$

where X^μ is the set of all states that include μ. Analyzing a system *a posteriori*, $\sum_{X^\mu} E[\widetilde{A}(x_i)]Pr(x_i)=0$ only in the case of equal fractions, because there always exist a pair of states with the same μ, the same probability and symmetric distribution around zero. The larger the game, the larger possible disproportions of $E[\widetilde{A}(x_i)]$ between the reference and the real system provided that in a real system strategies are drawn from flat distribution. As a result, $\sum_{X^\mu} E[\widetilde{A}(x_i)]Pr(x_i)$ grows with the population size. Hence, H_A as a function of the control parameter $n=N/P$ is larger than zero in the herd regime, if a system is different than the reference.

4.3.3 Variance per capita σ^2/N

For simplicity we consider here only the case of equal fractions. Hence we do not need to distinguish between *a priori* and *a posteriori*. The variance per capita is defined using Eq. (6). If game is large enough then a suitable approximation based on the MP representation is given by

$$\sigma^2(A) \simeq \sum_{i=1}^{\#X} Pr(x_i)E\big[A(x_i)\big]^2 \tag{26}$$

$$= \sum_{i=1}^{\#X} Pr(x_i)\bigg(\frac{N}{G}E\Big[\big(\sum_{\nu=1}^{G} C_\nu(x_i)\big)\Big]\bigg)^2 \tag{27}$$

The above approximation is based on replacing many potentially different values of $A(x_i)$ by their expected values. Generally, the approximation improves as number of agents increases. The number of fractions where all strategies suggest the same action after given μ is always $(2^{P-1})^S$ where 2^{P-1} represents the half of the strategy space where all strategies suggest the same action. Hence, at least, $2(2^{P-1})^S$ terms in C_ν in the sum compensate mutually. By that means there is $2^{PS}-2^{(P-1)S+1}$ terms that in the worst case are not compensated. Indeed, one can find states x where all actions of these fractions are equal to 1 or -1 but one can also find states where contributions of all fractions compensate to 0. Hence

$$0 \le |\sum_{j=1}^{G} C_\nu(x_i)| \le 2^{PS}-2^{(P-1)S+1} \tag{28}$$

The right-hand boundary can be redefined using $G=2^{PS}$ as $G(1-2^{1-S})$. E.g. for $S=2,3,4,5$ we have $\frac{G}{2}$, $\frac{3}{4}G$, $\frac{7}{8}G$, $\frac{15}{16}G$ respectively. Generally:

$$E[A(x_i)] \sim N \tag{29}$$

As a result in Eq. 26, $\sigma^2 \sim N^2$ and $\sigma^2/N \sim N$, in agreement with numerical simulations and theoretical results [10]. Variance stops to be proportional to N^2 if game leaves the herd regime. In the random mode, there is less agents than fractions and hence:

$$\sigma^2(x_i) = \sum_{n=1}^{N} Var[a_{\alpha'_n}(x_i)] \tag{30}$$

Considering further the case $S = 2$, on average the half of agents do not have choice because have two strategies that suggest the same action. Decisions of mentioned half population compensate mutually and does not influence A. There are states where the rest of the population has a choice and thus $\sigma^2(x_i) = \sum_{n=1}^{N/2} Var[a_{\alpha'_n}(x_i)]$. Hence,$\sigma^2 \sim \frac{N}{2}$.

In the cooperation regime most of fractions are in game but fluctuations of F are relatively large. Thus, there are fractions more and less populated. Strategies that are in less populated fractions win more frequently. Impact of these fractions is compensated by larger fractions and the variance is minimal. It reflects the balance between the crowd and anticrowd in the so called crowd-anticrowd approach [10].

4.4 The payoff $g(x) = x$

The case of linear payoff $g(x) = x$ requires different methods of analysis than the steplike one. For $g(x) = sgn(x)$ in every state there is a group of strategies that varies in suggested action but is characterized by the same utility. Hence, if an agent has two or more best strategies then he chooses one randomly. As a result some transitions are stochastic. The more so, the utility is bounded from the bottom and top: $U_{min} \leq U(t) \leq U_{max}$, where $U_{min(max)} = -(+)2^m$. As a result, the set of utility values is relatively small and the definition of the state can be based on these values. For $g(x) = x$, the probability that the pairwise different strategies have the same utility is unlikely compared to the case of $g(x) = sgn(x)$ and the range of possible U is much wider i.e. from $-N/2$ up to $N/2$, provided that the system is a reference one. As a result, stochasticity of transitions disappears nearly completely [12]. By that means a different definition of the state is required.

4.4.1 States of the Markov Chain

In any step of the game one can rank all pairwise different strategies as the best, 2nd best, 3rd best, etc. The size of the population that acts according to each of strategies is known [10],[12]. Hence, such ordered list of subscripts of different strategies, extended by μ value, is enough to fully describe the game behavior at a given moment. Thus, it can be considered as a state. In order to introduce the formal definition assume that $\{\beta_\iota\}_{\iota=1}^{2^P}$ is the set of pairwise different strategies indexed arbitrarily. There is sorting operator $\delta(\iota) \to \kappa$ that orders the strategies i.e. κ is the position of the strategy β_ι on the ordered list. Then the state is as follows.

$$x(t) = [\,\mu(t), \kappa_{\beta_1}(t), \ldots, \kappa_{\beta_{2^P}}(t)\,]. \tag{31}$$

The total number of states is equal to $\prod_{\iota=0}^{2^P/2-1}(2^P - 2\iota)$ and reflects all possible orders of P strategies provided that each strategy has its anti-strategy[3].

4.4.2 The initial phase

All steps where there are more then one strategy characterized by the same utility are called initial ones. If $U(t = 0) = $ const for all strategies then the number of some initial steps is necessary to split all utilities of pairwise different strategies. Now we show that the minimal number of steps is 2^m and the maximal is 2^{m+1}.

Utilities that have the same values at time t can differ in time $t + 1$ by $2A(t)$ or remain the same. They differ where corresponding pairwise different strategies suggest opposite actions after $\mu(t)$. Hence, the shortest time to split utilities of all 2^P strategies is 2^m. Such scenario requires appearance of all possible histories μ without any repetitions.

If strategies react in such fashion that their utilities do not split from step t to $t+1$, then it means that the same μ appears twice. As a result, the strategies that won in step t have to loose in step $t+1$ due to the positive change of the utility and preference of these strategies by most of the population in time $t+1$. Thereby the sign of $A(t+1)$ changes, compared to the sign of $A(t)$, and different μ has to appear. There is only one μ for which given half of different strategies reacts identically and for other μ split must appear. The example of both scenarios is presented if Fig. 7. The first scenario is relatively easy to follow and we pay attention only to the second one. The initial value is $\mu(0) = 1$ and all strategies have the same utility $U = 0$. Each agent at $t = 0$ draws one strategy randomly. Let us assume that most of them decide to use strategy suggesting $a(0) = -1$. As a result (i) α_2 and α_4 get positive payoff and, (ii) the next history is $\mu = 1$. After $\mu = 1$ both winning strategies suggest the same action and lose. As a result next history is $\mu = -1$. As history changed, the splited strategies have to react differently. It is because two different μ cannot cause the same reaction of all of strategies. Thus, the longest time to split all U trajectories is 2^{m+1} and requires every possible history to appear twice.

4.4.3 The order of states

The case of equal fractions: As we pointed out in Sec. 4.2, rewards and penalties have to compensate if

[3] First arbitrary chosen strategy from the set of 2^P strategies can be placed on one of 2^P positions on the ordered list. When the position is chosen then position for its anti-strategy is chosen automatically. Next, the strategy from the reduced set of $2^P - 2$ strategies is placed on one of $2^P - 2$ positions, and so forth.

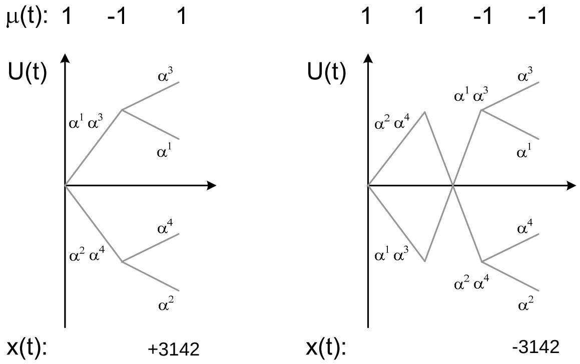


Figure 7: The shortest (left) and longest (right) scenario of the initial phase for $m = 1$ and $\mu(0) = 1$.

the game is stable. This requires specific order of states (cycle), such that every μ has to appear twice over the cycle to assure an appearance of reward and penalty that are of the same magnitude for any strategy. Such cycles are considered as attractors because, as we see, they attract all other initial states. The question is how many attractors exist and how one can find them. At least two ways of attack are possible provided that fractions are equal. The first case corresponds to a brute force method where for each state its successor is determined. But the usefulness of this method is limited only to small m. Another approach requires analysis of the Euler paths on de Brujin graph and is applicable for the whole range of m. We will show further that the number of attractors is two times larger than the number of Euler paths. Below are examples of both methods.

We present the brute force method for $m = 1$ and strategies defined as in Fig.8. Each state has to be analyzed and its successor has to be found. Fig. 9 presents relations between states. For simplicity, we use shorter notation for the state if $m = 1$ e.g. $[-1, 3, 4, 1, 2]$ maps for -3412. There are two attractors:

μ	α^1	α^2	α^3	α^4
-1	-1	-1	1	1
1	-1	1	-1	1

Figure 8: Four possible strategies for $m = 1$

$$\text{Attractor 1} = [+4231, -3142, -1324, +3142] \quad (32)$$

$$\text{Attractor 2} = [+4321, -3412, -1234, +3412] \quad (33)$$

Both attractors are equally possible provided that $U(t = 0) = \text{const}$ for all strategies. Each attractor assures that each possible history appears twice. One of appearances causes reward for half of strategies and another appearance causes penalty. Reward and penalty are of the same magnitude. The move along attractors assures that game follows Euler trail in the de Bruijn graph what is consistent with results of ref. [?][2]. The example of U trajectories that correspond to these attractors are presented in Fig. 10. The absolute changes of utilities are equal to one of two values: $N/2$ or $N/4$, depending on a state. In the former case, both best strategies suggest the same action. By that means the 3/4 of population acts according to these actions and an aggregate demand is equal to $|A| = \frac{3}{4}N - \frac{1}{4}N = \frac{N}{2}$. In the latter case, the first and the third strategy suggest the same action. Hence, the $\frac{10}{16}$ of the population chooses the same action and consequently $|A| = \frac{10}{16}N - \frac{6}{16}N = \frac{N}{4}$. Exemplified realization for $m = 1$ is shown in Fig. 11. It is seen that both distributions, A and $\text{sgn}(A)$, are symmetric. Since the game is fully deterministic, each of four states $x_1 \dots x_4$ is related to only one value of $A(x)$. Hence, the A distribution has four peaks.

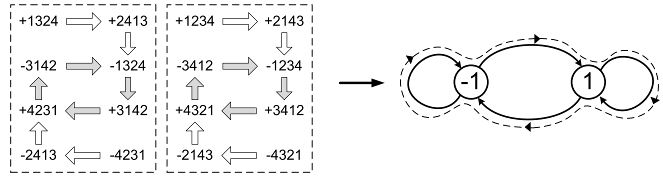


Figure 9: Two basins of attraction for $m = 1$ (left). Attractors are marked by gray arrows. Both attractors are projected to the same Euler path in de Bruijn graph (right).

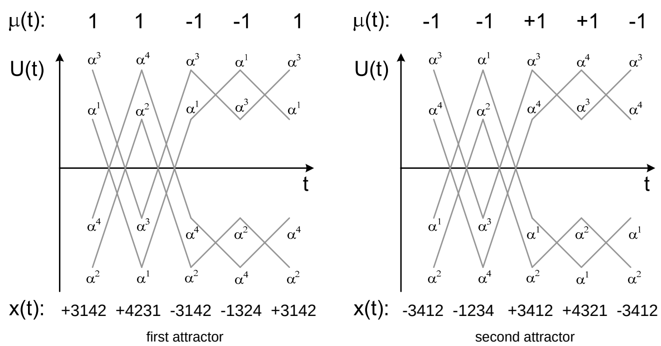


Figure 10: Utility trajectories for two possible attractors for $m = 1$.

More general way to determine the number of attractors is to count the number of Eulerian paths in de Bruijn graph. Each attractor consists of the unique set of states that do not appear in other attractors. We proved that every attractor comprises of exactly one state that is characterized by the large oscillation[4]: $|A| = N(1 - \frac{1}{2^{S-1}})$.

[4]A large oscillation is explicitly connected with a state that is characterized by half of best strategies suggesting the same action.

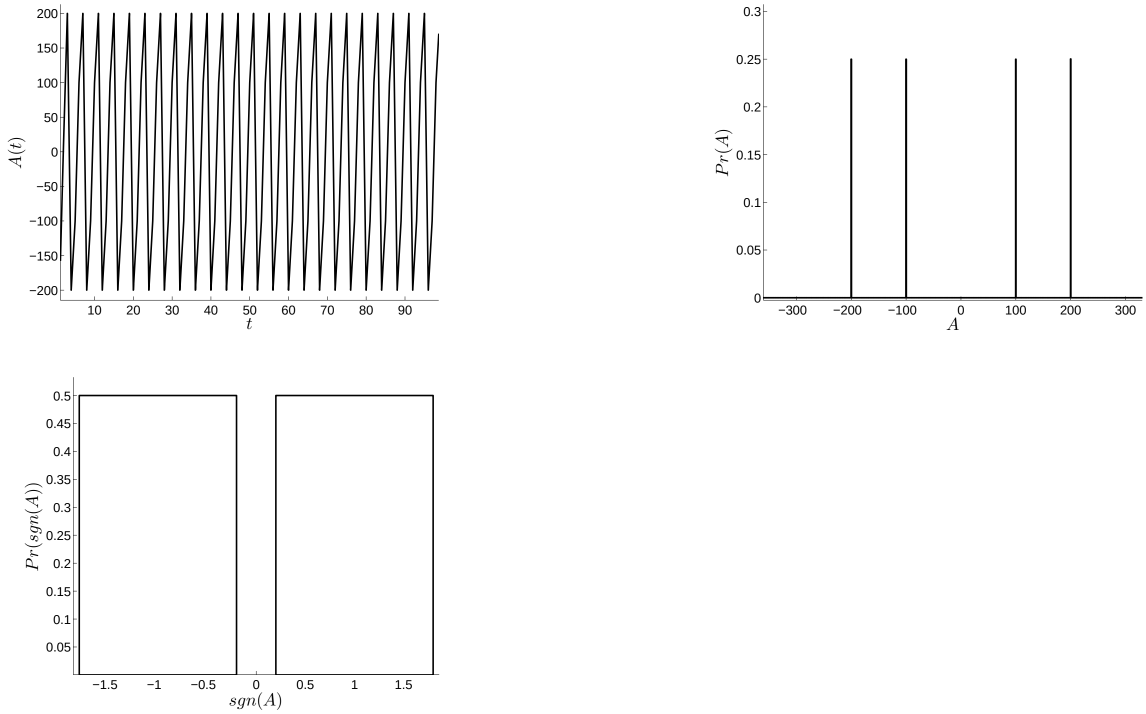


Figure 11: Time evolution of the aggregated demand $A(t)$, Estimated $Pr(A)$ and $Pr(\mathrm{sgn}(A))$ for the population size $N = 400$, agent memory $m = 1$, $s = 2$ strategies per agent, identical sizes of fractions (reference system) and linear payoff $g(x) = x$

The more so, the state has to incorporate the μ that represents one of the two possible homogenous nodes of the Bruijn graph [12]. As a consequence there are two different states that belong to two different attractors where both of attractors are projected on the same Eulerian path in de Bruijn graph. According to the theory of de Bruijn sequences, there is $2^{2^m}/2^{m+1}$ Eulerian paths. Hence, there is twofold attractors. E.g. there are $2, 4$ and 32 attractors for $m = 1, 2$ and 3.

The case of unequal fractions: The size of different fractions most likely varies if strategies are drawn randomly. This causes some shift of $\widetilde{A}$ distribution compared to the reference system. The mechanizm is the same as in the steplike payoff. As a result in each of states that belong to an attractor, the values of A are different compared to the case with equal fractions. If the game followed attractor the A would not compensate to zero along the path and the utility values would grow or shrink permanently. Nevertheless, the minority mechanizm stabilizes the game and prevents such scenario. This is reflected by additional states in the attractor. Exemplified realization for the case where strategies are drawn from uniform distribution is shown in Fig. 12. It is seen that both $Pr(\widetilde{A})$ and $Pr(\mathrm{sgn}(\widetilde{A}))$ are asymmetric. The comparison of Markov Chains where sizes of fractions are equal and different is shown in Fig. 13. It is seen that the game with unequal fractions mostly follows attractor 1 but in three of four states transitions to other states can appear either. The probability of these transitions is relatively small, what indicates that sizes of fractions are not far from the case where fractions are equal. The MP representation for unequal fractions is different for each realization.

4.4.4 The variance per capita σ^2/N

We proved in ref. [12] that large oscillations are periodic and equal to:

$$|A| = N\Big(1 - \frac{1}{2^{S-1}}\Big) \tag{34}$$

Hence, in particular, if $S = 2$, then $\sigma^2 \sim \frac{N^2}{4}$, is consistent with observations and results of ref. [10].

4.4.5 The stability and H behavior

The behavior of H_A is driven by absolute disproportions between fractions' sizes. The payoff is a direct function of A. Hence, to stabilize the game the negative and positive payoffs that follows the same μ have to compensate mutually. Hence, for any μ: $\langle A^-|\mu\rangle = \langle A^+|\mu\rangle$ and $\langle H_A\rangle = 0$. For this kind of payoff the same frequency of the negative and positive payoffs do not have to be preserved as it is required for $\mathrm{sgn}(x)$ (see Fig. 12, bottom left). The last point to understand is the plot of H_a/N that seems to be equal to zero in the herd regime. The H_a is the sum of $\langle a^*|\mu\rangle$ over P different μ's. Each of these components

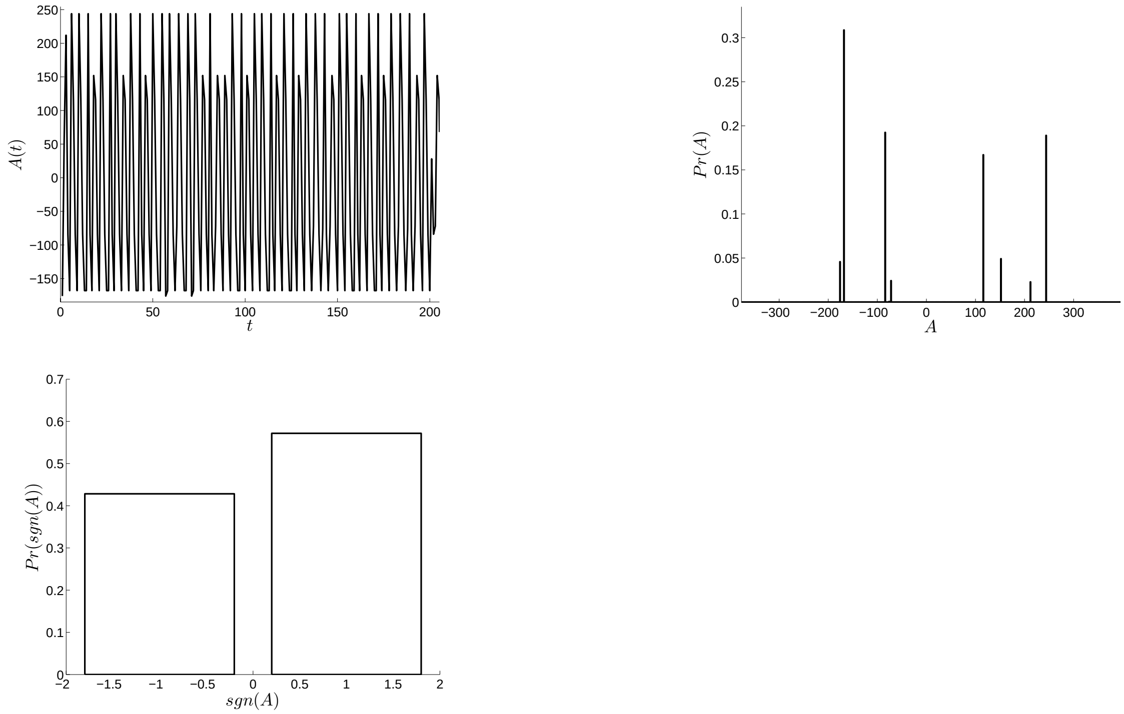


Figure 12: Time evolution of the aggregated demand $A(t)$, Plots of the estimated $Pr(\widetilde{A})$ and $Pr(\text{sgn}(\widetilde{A}))$ for the population size $N = 400$ and agent memory $m = 1$, $s = 2$ strategies per agent and unequal sizes of fractions.

is most likely nonzero and is bounded $\langle a^*|\mu\rangle = [-1, 1]$. Hence $\max(H_a) = \text{const} = P$ and in the limit $N \to \infty$ one has $H_a/N = 0$.

4.5 The influence of disproportions between fractions

One can try to measure how the size of disproportion between fractions influences transition probabilities in the Markov Chain. Hence, we introduce a measure D of the distance between two arbitrary processes. We mark the probabilities of former process by upper index R and latter by E what in our case reflects a reference system and an examined one. D is as follows.

$$\sum_{i \in E \bigcup R} \sum_{j \in E \bigcup R} \left| Pr^E(x_i) Pr^E(x_j|x_i) - Pr^R(x_i) Pr^R(x_j|x_i) \right| \tag{35}$$

The above measure is suitable to compare any MPs comprising even such where processes are based on different sets of states. The D is a real number $D \in [0, 2]$. If $D = 0$, then there are no differences between processes. If $D = 2$, then processes are based on strongly disjunctive sets of states. The standard deviation $\sigma(F)$ is a measure of disproportion of fractions. The distance as a function of $\sigma(F)$ is presented in Fig. 14. Left panel presents distance D measured between the reference MP (diagram 4, left) and 40 games where strategies are drawn from various distributions provided that payoff is $g(x) = \text{sgn}(x)$. In the case $g(x) = x$, the function is more complicated because we do not have just one MP that represents the reference system but two equiprobable attractors (two MPs). Therefore we use the sum of $D_1 + D_2$ as a function of disproportion of fractions as presented in Fig. 14 (right). If the game follows attractor 1 then $D_1 = 0$ and $D_2 = 2$.

5 Conclusions

This work aims to explain the behavior of macroscopic characteristics of MG: the variance per capita and predictability as a function of control parameter. An understanding of macroscopic phenomena requires analysis on deeper i.e. mesoscopic perspective. From this perspective agents' strategies that represent the same strategy in strategy space are treated as a whole what allows relatively easy to represent the game as a Markov Process. In this work we focus mostly on the herd regime especially on those games that fulfill $NS \gg 2^P$ requirement. It is the only case where such mesoscopic aggregations are possible. Two payoffs i.e steplike and linear are analyzed. We showed that in the case of the steplike payoff the stochastic and deterministic transitions are possible. In the case of the linear payoff transitions are deterministic. The stability mechanism is sensitive for the payoff function. When the steplike payoff is considered then the frequency of opposite signs of A after any μ has to be preserved. In case of linear payoff the negative and positive

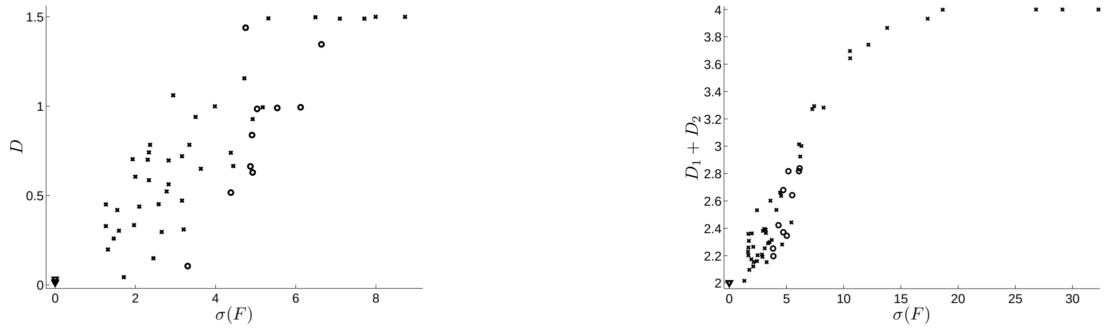


Figure 14: Distance D as a function of $\sigma(F)$ for two different payoffs $g(x) = \mathrm{sgn}(x)$ (left) and $g(x) = x$ (right).

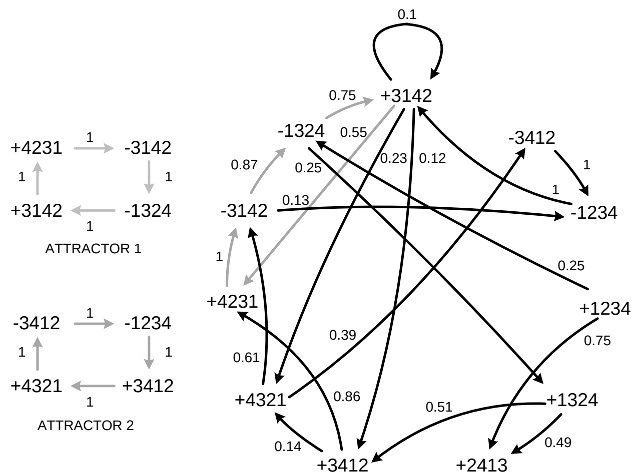


Figure 13: Two possible attractors for $m = 1$ (left) for game with equal sizes of fractions. The transition graph for a real game where sizes are unequal i.e. strategies are drawn from uniform distribution (right).

values of A have to compensate mutually. As a result, depending on the payoff, in the herd regime H_a or H_A is equal to zero. As the size of game grows then also the probability of differences between systems with equal fractions and unequal fractions arises what is particulary reflected (i) in distortions of transitions probability in the case of steplike payoff and (ii) in distortions of attractors in the case of linear payoff. To measure the size of distortion the distance between two MP is introduced.

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Appendix A: Deterministic transitions

Here, we show an example how to prove that the transition from a given state is deterministic provided that the system is a reference one. Additionally, we present that the transition can change if agents are assigned to fractions randomly. Consider the case $S = 2$. Fractions' indexes are assigned to each pair of strategies arbitrarily as follows: $F_1 : \{\alpha_1, \alpha_1\}$, $F_2\{\alpha_1, \alpha_2\}$, $F_3\{\alpha_1, \alpha_3\}$, $F_4\{\alpha_1, \alpha_4\}$, $F_5\{\alpha_2, \alpha_1\}$, $F_6\{\alpha_2, \alpha_2\}$, $F_7\{\alpha_2, \alpha_3\}$, $F_8\{\alpha_2, \alpha_4\}$, $F_9\{\alpha_3, \alpha_1\}$, $F_{10}\{\alpha_3, \alpha_2\}$, $F_{11}\{\alpha_3, \alpha_3\}$, $F_{12}\{\alpha_3, \alpha_4\}$, $F_{13}\{\alpha_4, \alpha_1\}$, $F_{14}\{\alpha_4, \alpha_2\}$, $F_{15}\{\alpha_4, \alpha_3\}$, $F_{16}\{\alpha_4, \alpha_4\}$. Let us consider an arbitrary chosen state were the transition is deterministic e.g. x_5 defined as $x_5 = [-1, 0, -2, 2, 0]$. Analyzing each fraction separately one finds that:

- For fractions $F_{11}, F_{12}, F_{15}, F_{16}$ both strategies suggest +1. Hence $C^\nu(x_i) = +1$, for $\nu \in \{11, 12, 15, 16\}$.

- For fractions $F_3, F_7, F_8, F_9, F_{10}, F_{14}$ strategy with higher U suggest +1. As a result for these strategies $C^\nu(x_i) = +1$, for $\nu \in \{3, 7, 8, 9, 10, 14\}$.

- In fractions F_1, F_2, F_5, F_6 both strategies suggest -1. Thus $C^\nu(x_i) = -1$, for $\nu \in \{1, 2, 5, 6\}$.

- Finally, fractions F_4, F_{13} have two strategies with equal probabilities but suggesting opposite actions. Hence, $C^\nu(x_i)$, for $\nu \in \{4, 13\}$, follows binomial distribution as expressed from formula (17).

For the reference system (equal fractions) one can calculate $E[A(x_5)] = \frac{5}{16}N$. The uncertainty is introduced by agents belonging to fractions F_4, F_{13} because they choose -1 or $+1$ with the same probability. It means that $A(x_5) \in \{\frac{3}{16}N \ldots \frac{5}{16}N\}$. Hence, $A(x_5)$ is always positive and $a^*(x_5) = -1$, thus the successor state is determined deterministically.

If agents are assigned to fractions randomly then each F is a random variable. Considering system *a priori* the expected value $E[A(x_5)]$ remains the same but the variance changes distinctly enough to allow for appearance of negative samples. If considering *a posteriori* also $E[\widetilde{A}(x_5)]$ is most likely biased compared to $E[A(x_5)]$.

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Appendix B: Symbol captions

$\langle\cdot\rangle = \frac{1}{T}\sum_{t=0}^{T}\cdot(t)$	– time average operator
a_α	– action suggested by strategy α
a^*	– the minority action
α_n^s	– the s-th strategy of the n-th agent
α_n'	– the active strategy, or the strategy of the highest utility, for the n-th agent
$A = \sum_{n=1}^{N} a_{\alpha_n'}$	– aggregated demand
$\{\beta_\iota\}_{\iota=1}^{2^P}$	– set of 2^P pairwise different strategies
C_ν	– normalized action of fraction ν
$\delta(\iota)$	sorting operator that orders the strategies β_ι
Δ_n	– set of S strategies of the n-th agent
D	– measure of the distance between two arbitrary processes
$E[\cdot]$	– expected value operator over possible realizations
F_ν	– size of fraction ν
f	– frequency of demand peaks
g	– payoff function
G	– number of different fractions
H_a, H_A	– predictabilities
κ	– position of the strategy β_ι on the ordered list
m	– length of the sequence of last minority decisions
$\mu = [a^*(t-m), \ldots, a^*(t-1)]$	– sequence of the last minority decisions
N	– the total number of agents in the game
O, Q, R	– number of fractions for which C is $+1$, -1 or within $[-1, 1]$
$P = 2^m$	– number of possible realizations of μ
$Pr(x_j \vert x_i)$	– transition probability from state x_i to x_j
$Pr(x_i)$	– probability of being in state x_i
$\rho(n)$	– distribution of strategies for the n-th agent at the beginning of the game
R_α	– payoff for the strategy α
$\sigma^2(A) = \frac{1}{T}\sum_{t=0}^{T} A(t)^2$	– mean square value over the time
S	– the total number of strategies for each agent
U_α	– utility of the strategy α
$U_{min(max)} = -(+)2^m$	– the absolute minimum (maximum) value of the utility
$x(t) = [\,\mu(t), U_1(t), \ldots, U_{2^P}(t)\,]$	– state of the game at time t for the steplike payoff
$x(t) = [\,\mu(t), \kappa_{\beta_1}(t), \ldots, \kappa_{\beta_{2^P}}(t)\,]$	– state of the game at time t for the linear payoff
$Var[\cdot]$	– variance operator over possible realizations
$\#Z$	– number of elements of Z