Autonomous Dynamical Systems with Periodic Coefficients

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Summary. The solutions of autonomous dynamical systems with periodic coefficients mainly depend on the Floquet-Liapunov exponents of a Hill's associated equation. These exponents are computed without integration by a very fast algorithm which exponentially converges.

So, some important features of the solutions behaviour, such as the location of the temporal mean in the phase plane, funnelling phenomenon, period doubling, parametric resonances, can be specified. In this paper, the implementation of the method is shown on a parametric Van der Pol equation.

Key words: slow-fast dynamics; Floquet-Liapunov exponents; Hill's equation.

1 Some previous results

Consider the Van der Pol equation:

$$\begin{cases} \varepsilon \frac{dx(t)}{dt} = -\frac{x^3}{3} + x + y \\ \frac{dy(t)}{dt} = -x \end{cases} \tag{1}$$

where $\varepsilon = 0.05$. In previous works, we established that the location of the points where the curvature of the trajectory in the phase plane vanishes provides the following equation of a slow manifold [1]:

$$\phi = y - \frac{x^5 - 4x^3 + 3x\,(1-\varepsilon)}{3\,(x^2-1)} = 0 \tag{2}$$

Moreover, the attractive part of this manifold, given by $\mathbf{V}.\boldsymbol{\nabla}\phi > 0$, is an invariant manifold.

2 Interpretation

Let $\xi(t)$ and $\eta(t)$ be small variations around any point (x, y). The variation equation, which we call "tangent-system", is the linearized equation:

$$\begin{bmatrix} \frac{d\xi(t)}{dt} \\ \frac{d\eta(t)}{dt} \end{bmatrix} = \begin{bmatrix} \frac{\partial f(x,y)}{\partial x} & \frac{\partial f(x,y)}{\partial y} \\ \frac{\partial g(x,y)}{\partial x} & \frac{\partial g(x,y)}{\partial y} \end{bmatrix} \begin{bmatrix} \xi(t) \\ \eta(t) \end{bmatrix} = \begin{bmatrix} \frac{1}{\varepsilon}\left(-x^2+1\right) & \frac{1}{\varepsilon} \\ -1 & 0 \end{bmatrix} \begin{bmatrix} \xi(t) \\ \eta(t) \end{bmatrix} \quad (3)$$

If $\lambda_k, k = 1, 2$, are the eigenvalues, the eigenvectors are given by :

$$y_k = \begin{bmatrix} -\lambda_k \\ 1 \end{bmatrix}, \quad k = 1, 2$$

Each eigenvalues determine a mode. In the regions of the phase plane where the eigenvalues are real, one of them, say λ_2, is proportional to ε^{-1} and its absolute value much larger. When it is negative, the associated mode is evanescent. Let λ_1 be the other eigenvalue. We also established that the exact equation of the invariant slow manifold, on which the slow trajectories of the dynamical system (1) lean, is obtained by writing that the velocity is parallel to the eigendirection y_1 associated to the non evanescent mode. So, we obtain another expression of the exact equation of the slow part of the Van der Pol equation limit cycle:

$$y = -\frac{x^3}{3} - x + \varepsilon\ x\ \lambda_1 = \frac{x^3}{3} - x + \frac{1}{2}x\ \left(1 - x^2 + \sqrt{(1-x)^2 - 2\varepsilon}\right) \quad (4)$$

3 Autonomous dynamical systems with periodic coefficients

Now, consider the parametric Van de Pol equation

$$\begin{cases} \frac{dx(t)}{dt} = \frac{1}{\varepsilon}\left(-\frac{x^3}{3} + x + \theta(t)y\right) \\ \frac{dy(t)}{dt} = -x \end{cases} = \begin{bmatrix} f(x,y,t) \\ \\ g(x,y,t) \end{bmatrix} \quad (5)$$

where $\theta(t)$ is a T-periodic mapping from $\mathbb{R}$ to $\mathbb{R}$, $t \mapsto \theta(t)$, supposed to have a Fourier development limited to p_M terms :

$$\theta(t) = \sum_{p=-p_M}^{p_M} \theta_P e^{ip\omega t}, \text{ with } \omega = \frac{2\pi}{T} \text{ and } \theta_{-k} = \theta_k^* \quad (6)$$

Let $\theta_0 = 1$. Now, the variation equation is a homogeneous linear equation with periodic coefficients, a Hill equation:

$$\begin{bmatrix} \frac{d\xi(t)}{dt} \\ \frac{d\eta(t)}{dt} \end{bmatrix} = \begin{bmatrix} \frac{1}{\varepsilon}\left(-x^2+1\right) & \frac{\theta(\mathrm{t})}{\varepsilon} \\ -1 & 0 \end{bmatrix} \begin{bmatrix} \xi(t) \\ \eta(t) \end{bmatrix} \quad (7)$$

According the Floquet theory, each of the two solutions of this equation can be written as:

$$\eta\left(t\right) = e^{\mu t}\psi\left(t\right), \quad (8)$$

where the complex number μ is the Floquet exponent and $\psi\left(t\right)$ the T-periodic part of the solution

$$\psi\left(t\right) = \sum_{m=-\infty}^{\infty} \psi_m e^{im\omega t}, \text{ with } \psi_{-m} = \psi_m^*. \quad (9)$$

There are two modes corresponding to the two Floquet exponents. We use the same argument as previously: if the fast Floquet exponent has a real negative part, the associated mode is evanescent and the other mode, a slow solution, remains.

4 Floquet's exponents computing

A generalization of the G. W. Hill method [4] allows us to deduce a very fast algorithm that exponentially converges to the exact numerical value [5].

5 Behaviour of the temporal mean of the solutions

Proposition 5.1 *Let μ_1 the slow Floquet's exponent. We introduce the associated dynamical system as the dynamical system with constant coefficients having μ_1 as slow eigenvalue. The temporal means of the slow solutions of a parametric dynamical system are located on the slow curve of this associated system. In our example, the equation of this curve is given by:*

$$y = -\frac{x^3}{3} - x + \varepsilon x \mu_1 \quad (10)$$

Proof. According to the Floquet's theorem, let $\eta\left(t\right) = e^{\mu_1 t}\psi\left(t\right)$be the particular solution of (6) associated to μ_1, the slow value of μ, where $\psi\left(t\right)$ is a T-periodic mapping. Since $\psi\left(nT\right) = \psi\left(0\right)$, $\forall n$ we have $\eta\left(nT\right) = \eta\left(0\right)e^{\mu_1 nT}$. The temporal means are located on the eigendirection related to the slow solutions of a dynamical system having $\mu_1$as slow eigenvalue. Then the trajectories are given by (4), in which λ_1 is replaced by μ_1.

6 Some considerations on stability

Proposition 6.1 *Let $\phi(x,y) = y - \frac{x^3}{3} + x - \varepsilon\ x\ \mu_1 = 0$ the equation of the curve where the means are located. This curve is attractive with respect to the trajectories of the system if the scalar product $\mathbf{V}.\boldsymbol{\nabla}\phi$ is positive, where v is the velocity. The values where the stability changes, corresponding to $\mathbf{V}.\boldsymbol{\nabla}\phi > 0$, are*

$$\mathbf{V}.\boldsymbol{\nabla}\phi(x,y) = \left(-\frac{x^3}{3} + x + \theta(t)y\right)\left(-x^2 + 1 - \varepsilon\mu_1\right) - \varepsilon x = 0 \qquad (11)$$

$$\textit{where} \quad \theta(t)\,y = \frac{x^3}{3} - x + \frac{\varepsilon x}{-x^2 + 1 - \varepsilon\mu_1} \qquad (12)$$

7 Funnelling

Proposition 7.1 *The amplitude of the oscillations of the slow mode decreases in the regions where the Floquet exponent μ_1 is negative. It is sensitive to initial conditions in the regions where μ_1 is positive.*

Proof. The amplitude of the oscillations of the slow mode being modulated by $e^{\mu_1 t}$, their amplitude decreases all along the mean curve where μ_1 is negative and exponentially increases all along the mean curve where μ_1 is positive. We have drawn the phase plane for several values of the amplitude and of the frequency for a sinusoidal parameter $\theta(t)$. In every picture, this funnelling phenomenon appears along the slow trajectories.

8 Parametric resonance, period doubling

The imaginary part of the Floquets exponent is related to the frequency of oscillations: some important features of the solution, such as parametric resonances and bifurcations by period doubling can be brought to light.

9 Discussion

Several features, like the mean location, the amplitude, the stability of solutions of parametric dynamical systems depend on the real part of the Floquet coefficients. The imaginary part is related to parametric resonances and bifurcations by period doubling. The same algorithm can be used to compute the Liapunov exponents of nth- order Hill's equations associated to higher order parametric dynamical systems, for example predator-prey models that take in account the daily or the annual variations of parameters, or dynamical systems related to periodic biologic rythms.

10 Figures

In this section are presented some solutions of the Van der Pol parametric equation

$$\begin{cases} \frac{dx(t)}{dt} = \frac{1}{\varepsilon}\left(-\frac{x^3}{3} + x + \theta(t)y\right) \\ \frac{dy(t)}{dt} = -x \end{cases}$$

With $\theta\left(t\right) = \theta_0 + \theta_1 Cos\left(\omega t\right)$. For all pictures, $\varepsilon = 0.05$ and $\theta_0 = 1$. The period T and the amplitude θ_1 of the only harmonic are specified. The red line is the mean trajectory of the parametric dynamical system, say the invariant manifold of the associate dynamical system having the Floquet's exponent as eigenvalue. The green line is the nullcline, corresponding to $\frac{dx(t)}{dt} = 0$ and $\theta\left(t\right) = \theta_0 = 1$. The blue line is the invariant manifold of the dynamical system with the constant coefficient $\theta_0 = 1$, the mean of the periodic coefficient.

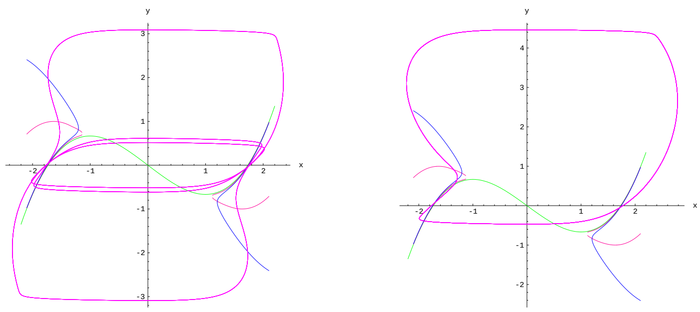

Fig. 1. Amplitudes 1, 1.5 - Period 5

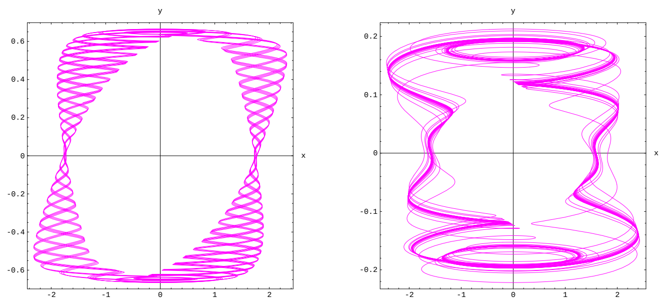

Fig. 2. Amplitudes 4, 30 - Periods 0.1, 0.089493

Table 1. The Floquet - Liapunov exponents role

	Tangent Linear System	**Hill System**
Characteristic exponent	Eigenvalue	Floquet exponent
Symbol	λ	μ
Fast evanescent modes	Large and negative values of $Re\,(\lambda)$	Large and negative values of $Re\,(\mu)$
Slow remaining modes	Smaller values of $Re\,(\lambda)$	Smaller values of $Re\,(\mu)$
Manifold $\phi = 0$ in the phase plane	The location of the points where trajectories are parallel to a slow eigendirection defines the slow manifold $\phi\,(\lambda) = 0$	To define the trajectory replace λ by μ in $\phi\,(\lambda)$. So we obtain $\phi\,(\mu) = 0$
Attractive part of the manifold: $\phi = 0$ **and** $\mathbf{V}.\nabla\phi < 0$	Invariant manifold	Domain of the phase plane where the amplitude of the oscillation decreases (funnelling)
Repulsive part of the manifold: $\phi = 0$ **and** $\mathbf{V}.\nabla\phi > 0$	If there is a fast positive eigenvalue: unreachable part of the manifold If they are all negative: weakly repulsive part of the manifold	Domain of the phase plane where the amplitude of the oscillation increases
Imaginary part	Frequency of oscillation, period doubling	Frequency of oscillation: period doubling (bifurcations) and parametric resonance

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